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Journal of Geometry and Physics 31 (1999) 111–141

JOURNAL OF GEOMETRY AND PHYSICS

# Algebroids – general differential calculi on vector bundles ★

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Received 29 September 1998; received in revised form 5 January 1999

**Abstract**

A notion of an algebroid – a generalization of a Lie algebroid structure on a vector bundle is introduced. We show that many objects of the differential calculus on a manifold $M$ associated with the canonical Lie algebroid structure on $\mathsf{T}M$ can be obtained in the framework of a general algebroid. Also a compatibility condition which leads, in general, to a concept of a bialgebroid.


*Subj. Class.:* Differential geometry
*1991 MSC:* 17B66; 58F05; 58H05
*Keywords:* Lie algebroid; Exterior derivative; Poisson structure; Tangent lift; Vector bundle; Lie bialgebra

## 0. Introduction

The classical Cartan differential calculus on a manifold $M$, including the exterior derivative d, the Lie derivative $\mathcal{L}$, etc., can be viewed as being associated with the canonical Lie algebroid structure on $\mathsf{T}M$ represented by the Lie bracket of vector fields. Lie algebroids have been introduced repeatedly into differential geometry since the early 1950s, and also into physics and algebra, under a wide variety of names. They have been also recognized as infinitesimal objects for Lie groupoids [18]. We refer to [14] for basic definitions, examples, and an extensive list of publications in these directions.

★ Supported by KBN, grant no. 2 PO3A 074 10.
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PII: S0393-0440(99)00007-8

Being related to many areas of geometry, like connection theory, cohomology theory, invariants of foliations and pseudogroups, symplectic and Poisson geometry, etc., Lie algebroids became recently an object of extensive studies.

What we propose in this paper is to find out what are, in fact, the structures responsible for the presence of a version of the Cartan differential calculus on a vector bundle and how are they related. This leads to the notion of a general algebroid.

It is well known that there exists a one–one correspondence between Lie algebroid structures on a vector bundle $\tau : E \to M$ and linear Poisson structures on the dual vector bundle $\pi : E^* \to M$. This correspondence can be extended to much wider class of binary operations (brackets) on sections of $\tau$ on one side, and linear contravariant 2-tensor fields on $E^*$ on the other side. It is not necessary for these operations to be skew-symmetric or to satisfy the Jacobi identity. The vector bundle $\tau$ together with a bracket operation, or the equivalent contravariant 2-tensor field, will be called an algebroid. This terminology is justified by the fact that contravariant 2-tensor fields define certain binary operations on the space $C^\infty(E^*)$. The algebroids constructed in this way include all finite-dimensional algebras over real numbers (e.g. associative, Jordan, etc.) as particular examples. The base manifold $M$ is in these cases a single point.

Searching for structures which give us differential calculi on vector bundles, we look at objects of analytical mechanics as related to the Lie algebroid structure of the tangent bundle.

The tangent bundle $\mathsf{T}M$ is the canonical Lie algebroid associated with the canonical Poisson tensor (symplectic form) on $\mathsf{T}^*M$. Other canonical objects associated with $\mathsf{T}M$ are: the canonical isomorphism

$$\alpha_M : \mathsf{T}\mathsf{T}^*M \longrightarrow \mathsf{T}^*\mathsf{T}M$$

of double vector bundles, discovered by Tulczyjew [19], dual to the well-known flip

$$\kappa_M : \mathsf{T}\mathsf{T}M \longrightarrow \mathsf{T}\mathsf{T}M,$$

and the tangent lift $\mathrm{d}_\mathsf{T}$ of tensor fields on $M$ to tensor fields on $\mathsf{T}M$ (cf. [5,17,22]).

The algebroid structure of the tangent bundle is not used in analytical mechanics directly via its Lie bracket of vector fields. In the Lagrange formulation of the infinitesimal dynamics one uses the mentioned isomorphism $\alpha_M$. The canonical symplectic structure is the basic for the Hamiltonian mechanics, but to obtain Euler–Lagrange equation we use the complete tangent lift $\mathrm{d}_\mathsf{T}$. Following Weinstein [21] and Libermann [12] also Lie algebroids other than that of the tangent bundle can be used in the variational formulation of the dynamics. It is important to know if one can characterize an algebroid structure directly in terms of objects like $\mathrm{d}_\mathsf{T}$, $\alpha_M$, etc.

The search for a basis of the Cartan differential calculus on a vector bundle is not the only reason for studying general algebroids. We are motivated also by the fact that the interest in Lie-like, but not skew-symmetric, or not satisfying the Jacobi identity, structures has been growing in last years.

For example, structures more general than Lie bialgebras appear as semi-classical limits of quasi-Hopf algebras [3]. The general theory of such objects, known under the name of Jacobian quasi-bialgebras, was developed by Kosmann-Schwarzbach [9]. They form the infinitesimal part of quasi-Poisson Lie groups. The dual objects are Poisson quasi-groups, so that their infinitesimal parts are quasi-Lie algebras, i.e. 'not quite Lie algebras'. We are sure all this has an algebroid counterpart in the spirit of our work. Some examples are sketched in Section 9.

Also in the theory of webs one can find objects similar to Lie groupoids but not associative in general [4]. We think that their infinitesimal parts are algebroids in our sense.

Let us also mention the concept of Loday algebras, i.e. Leibniz algebras in the sense of Loday [13] (cf. also [10]) which are 'non-skew-symmetric Lie algebras'.

Finally, let us recall the use of Nijenhuis tensors in defining deformed Lie algebroid structures on tangent bundles. Considering properties of $(1, 1)$-tensors less restrictive than vanishing of the Nijenhuis torsion, we get algebroids in the sense we propose.

All this provides another motivation for our work.

The paper is organized as follows.

In Sections 1–3 we show that objects similar to $\alpha_M$, $\kappa_M$, and $\mathrm{d}_\mathsf{T}$ can be associated with any algebroid structure. Each of these objects separately provides a complete description of the algebroid structure. Other constructions, known for Lie algebroids are extended to general algebroids. We define the tangent and the cotangent lifts of an algebroid in Section 4 and the Lie derivative in Section 5. We find an operation which acts as the exterior derivative in a limited way. Conditions for an algebroid to be a Lie algebroid are formulated in terms of these objects (Theorems 8 and 10).

The discussion of algebroids is extended to bialgebroids in Section 6. Alternative definitions of bialgebroids are considered. A link to the original definition of Mackenzie and Xu [15] is established by Theorem 13.

In Sections 7 and 8 we discuss the algebroid constructions in the important case of an algebroid defined by a linear connection on the tangent bundle $\mathsf{T}M$. The Lie derivative for this algebroid coincides with the covariant derivative. The algebroid of a linear connection is not skew-symmetric and, consequently, is not a Lie algebroid.

The canonical example of an algebroid is the Lie algebroid $A(G)$ of a Lie groupoid $G$. The Lie algebroid structure on $A(G)$ is obtained from the canonical Lie algebroid structure of the tangent bundle $\mathsf{T}G$. Thus, one can expect that objects of the Lie algebroid $A(G)$ can be obtained directly from the corresponding objects of the canonical algebroid $\mathsf{T}G$. We show in Section 9 how the algebroid lift can be obtained from the tangent lift $\mathrm{d}_\mathsf{T}$ and we discuss also the relation between the lift $\mathrm{d}_\mathsf{T}$ and the lifting of multiplicative vector fields, described in [16]. All this can be done (at least partially) for pre- (or quasi- ; the terminology is not fixed yet) Lie groupoids $G$, which are structures weaker than Lie groupoids (we do not assume associativity). Having in mind some natural examples of this kind related to quasi-Poisson Lie groups, webs, quasi-Nijenhuis tensors, etc., we are planning to discuss these problems in a further publication.

### 0.1. Notation

Let $M$ be a smooth manifold. We denote by $\tau_M : \mathsf{T}M \to M$ the tangent vector bundle and by $\pi_M : \mathsf{T}^*M \to M$ the cotangent vector bundle.

Let $\tau : E \to M$ be a vector bundle and let $\pi : E^* \to M$ be the dual bundle. We use the following notation for tensor bundles:

$$\tau^{\otimes k} : E \otimes_M \cdots \otimes_M E = \otimes^k_M(E) \longrightarrow M,$$
$$\tau^{\wedge k} : E \wedge_M \cdots \wedge_M E = \wedge^k_M(E) \longrightarrow M,$$

the module of sections over $\mathcal{C}^\infty(M)$:

$$\otimes^k(\tau) = \Gamma(\otimes^k_M(E)), \qquad \Phi^k(\tau) = \Gamma(\wedge^k_M(E)),$$

and the corresponding tensor algebras

$$\otimes(\tau) = \oplus_{k\in\mathbb{Z}} \otimes (\tau)^k(E),$$
$$\Phi(\tau) = \oplus_{k\in\mathbb{Z}} \Phi^k(\tau),$$
$$\mathfrak{T}(\tau) = \oplus_{k\in\mathbb{Z}} \otimes^k (\tau \oplus \pi) = \oplus_{k\in\mathbb{Z}} \mathfrak{T}^k(\tau),$$

where

$$\tau \oplus \pi : E \oplus E^* \longrightarrow M$$

is the Whitney sum. In particular,

$$\otimes^0(\tau) = \Phi^0(\tau) = \mathfrak{T}^0(\tau) = \mathcal{C}^\infty(M)$$

and

$$\otimes^k(\tau) = \Phi^k(\tau) = \mathfrak{T}^k(\tau) = \{0\} \quad \text{for } k < 0.$$

By $\langle\cdot,\cdot\rangle$, we denote the canonical pairing between $E$ and $E^*$ as well as pairings between the corresponding tensor bundles, e.g.,

$$\langle\cdot,\cdot\rangle : \otimes^k_M(E) \times_M \otimes^k_M(E^*) \longrightarrow \mathbb{R},$$

and pairings of sections, e.g.,

$$\langle\cdot,\cdot\rangle : \otimes^k(\tau) \times \otimes^k(\pi) \longrightarrow \mathcal{C}^\infty(M).$$

Let $K$ be a section of a tensor bundle $\otimes^k_M(E)$, $K \in \otimes^k(\tau)$. We denote by $\iota(K)$ the corresponding linear function on the dual bundle

$$\begin{aligned} \iota(K) &: \otimes^k_M(E^*) \to \mathbb{R} \\ &: a \mapsto \langle K(m), a\rangle, \quad m = \pi^{\otimes k}(a). \end{aligned} \tag{1}$$

For a section $X$ of $\tau$ ($X \in \otimes^1(\tau)$), we have the usual operators of insertion

$$\begin{aligned} \mathrm{i}^l_X &: \otimes^{k+1}(\pi) \to \otimes^k(\pi) \\ &: \mu_1 \otimes \cdots \otimes \mu_{k+1} \mapsto \langle X, \mu_l\rangle \mu_1 \otimes \overset{l}{\overset{\vee}{\cdots}} \otimes \mu_{k+1}, \end{aligned} \tag{2}$$

where $\mu_j \in \otimes^1(\pi)$ and $\overset{l}{\vee}$ stands for the omission. We denote by $\mathrm{i}_X$ the derivation of the tensor algebra $\otimes(\pi)$:

$$\mathrm{i}_X(\mu_1 \otimes \cdots \otimes \mu_{k+1}) = \sum_{l=1}^{k+1} (-1)^{l+1} \mathrm{i}_X^l(\mu_1 \otimes \cdots \otimes \mu_{k+1}).$$

Similarly, for a section $K$ of $E^* \otimes_M E$, we have the derivation $\mathrm{i}_K$ of the full tensor algebra $\mathfrak{T}(\tau)$ determined by

$$\mathrm{i}_{\mu\otimes X}(Y) = -\langle \mu, Y\rangle X, \quad \mathrm{i}_{\mu\otimes X}(\nu) = \langle \nu, X\rangle \mu, \quad \mathrm{i}_{\mu\otimes X}(f) = 0$$

for $X, Y \in \otimes^1(\tau)$, $\mu, \nu \in \otimes^1(\pi)$, $f \in C^\infty(M)$.

Let $\Lambda \in \otimes^2(\tau)$. We denote by $\widetilde{\Lambda}$ the mapping

$$\widetilde{\Lambda} : E^* \to E, \quad \widetilde{\Lambda} \circ \mu = \mathrm{i}_\mu^1 \Lambda.$$

### 0.2. Local coordinates

Let $(x^a)$, $a = 1, \ldots, n$, be a coordinate system in $M$. We introduce the induced coordinate systems

$$(x^a, \dot{x}^b) \text{ in } \mathsf{T}M, \quad (x^a, p_b) \text{ in } \mathsf{T}^*M.$$

Let $(e_1, \ldots, e_m)$ be a basis of local sections of $\tau : E \to M$ and let $(e_*^1, \ldots, e_*^m)$ be the dual basis of local sections of $\pi : E^* \to M$. We have the induced coordinate systems:

$$\begin{aligned}
&(x^a, y^i), \quad y^i = \iota(e_*^i) \quad \text{in } E, \qquad (x^a, \xi_i), \quad \xi_i = \iota(e_i) \quad \text{in } E^*,\\
&(x^a, y^{i_1\cdots i_r}) \quad \text{in } \otimes_M^r E, \qquad (x^a, \xi_{i_1\cdots i_r}) \quad \text{in } \otimes_M^r E^*,
\end{aligned}$$

and

$$\begin{aligned}
&(x^a, y^i, \dot{x}^b, \dot{y}^j) \quad \text{in } \mathsf{T}E, \qquad (x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) \quad \text{in } \mathsf{T}E^*,\\
&(x^a, y^{i_1\cdots i_r}, \dot{x}^b, \dot{y}^{j_1\cdots j_r}) \quad \text{in } \mathsf{T}\otimes_M^r E,\\
&(x^a, y^i, p_b, \pi_j) \quad \text{in } \mathsf{T}^*E, \qquad (x^a, \xi_i, p_b, \varphi^j) \quad \text{in } \mathsf{T}^*E^*,\\
&(x^a, \xi_{i_1\cdots i_r}) \quad \text{in } \otimes_M^r E^*, \qquad (x^a, y^i, \pi_{\beta_1\cdots\beta_r}) \quad \text{in } \otimes_E^r \mathsf{T}^*E,
\end{aligned}$$

where $\beta_i \in \{1', \ldots, n'\} \cup \{1, \ldots, m\}$.

We have the canonical symplectic forms:

$$\omega_{E^*} = \mathrm{d}p_a \wedge \mathrm{d}x^a + \mathrm{d}\varphi_i \wedge \mathrm{d}\xi_i$$

on $\mathsf{T}^*E^*$ and

$$\omega_E = \mathrm{d}p_a \wedge \mathrm{d}x^a + \mathrm{d}\pi_i \wedge \mathrm{d}y^i$$

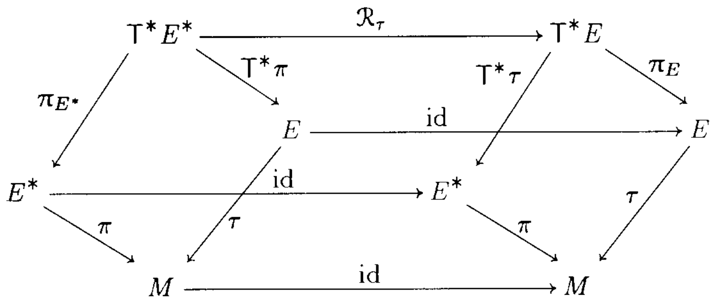
on $\mathsf{T}^*E$, and the corresponding Poisson tensors

$$\Lambda_{E^*} = \partial_{p_a} \wedge \partial_{x^a} + \partial_{\varphi^j} \wedge \partial_{\xi_j} \qquad \text{and} \qquad \Lambda_E = \partial_{p_a} \wedge \partial_{x^a} + \partial_{\pi_j} \wedge \partial_{y^j}.$$

There is also a canonical isomorphism (cf. [8])

$$\mathcal{R}_\tau : \mathsf{T}^*E^* \longrightarrow \mathsf{T}^*E \tag{3}$$

being an anti-symplectomorphism and also an isomorphism of double vector bundles




. (4)

In local coordinates, $\mathcal{R}_\tau$ is given by

$$(x^a, y^i, p_b, \pi_j) \circ \mathcal{R}_\tau = (x^a, \varphi^i, -p_b, \xi_j).$$

## 1. Leibniz structures and algebroids

**Definition 1.** A *Leibniz structure* is a pair $(M, \Lambda)$, where $M$ is a manifold and $\Lambda$ is a contravariant 2-tensor field. A Leibniz structure defines the *Leibniz bracket* $\{\cdot, \cdot\}_\Lambda$ on $C^\infty(M)$ by

$$\{f, g\}_\Lambda = \langle \Lambda, \mathrm{d}f \otimes \mathrm{d}g \rangle.$$

The bracket $\{\ ,\ \}$ is a bilinear operation satisfying the Leibniz rules

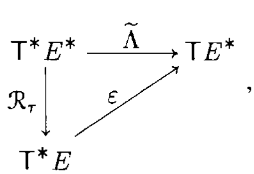
$$\{f, gh\}_\Lambda = \{f, g\}_\Lambda h + g\{f, h\}_\Lambda, \qquad \{fh, g\}_\Lambda = \{f, g\}_\Lambda h + f\{h, g\}_\Lambda. \tag{5}$$

A Leibniz structure is called *skew-symmetric* if the tensor $\Lambda$ or, equivalently, the bracket $\{\ ,\ \}_\Lambda$, is skew-symmetric.

**Remark.** A Leibniz bracket, which is skew-symmetric and satisfies the Jacobi identity is called a *Poisson bracket* and the corresponding tensor – a *Poisson structure*. It is well known that Lie algebroid structures on a vector bundle $E$ correspond to linear Poisson structures on $E^*$.

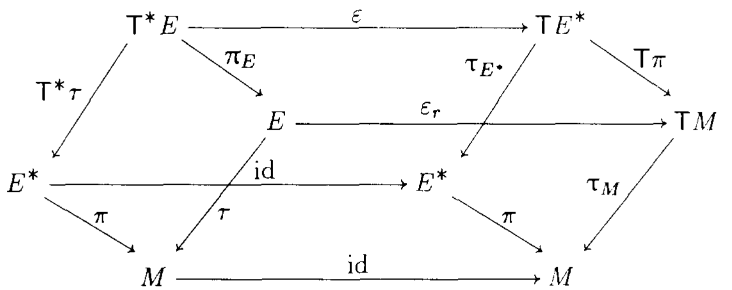
A Leibniz structure $\Lambda$ on $E^*$ is called *linear* if the corresponding mapping $\widetilde{\Lambda} : \mathsf{T}^*E^* \to \mathsf{T}E^*$ is a morphism of double vector bundles.

Let $\tau : E \to M$ be a vector bundle. The commutative diagram




, (6)

describes a one-to-one correspondence between linear Leibniz structures $\Lambda$ on $E^*$ and homomorphisms of double vector bundles (cf. [8])




(7)

The core of a double vector bundle is the intersection of the kernels of the projections. It is obvious that the core of $\mathsf{T}^*E$ ($\mathsf{T}E^*$) can be identified with $\mathsf{T}^*M$ ($E^*$). With these identifications the induced by $\varepsilon$ morphism of cores is a morphism

$$\varepsilon_c : \mathsf{T}^*M \to E^*.$$

In local coordinates, every $\varepsilon$ as in (7) is of the form

$$(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) \circ \varepsilon = (x^a, \pi_i, \rho_k^b(x)y^k, c_{ij}^k(x)y^i\pi_k + \sigma_j^a(x)p_a) \tag{8}$$

and corresponds to the linear Leibniz tensor

$$\Lambda_\varepsilon = c_{ij}^k(x)\xi_k\partial_{\xi_i} \otimes \partial_{\xi_j} + \rho_i^b(x)\partial_{\xi_i} \otimes \partial_{x^b} - \sigma_j^a(x)\partial_{x^a} \otimes \partial_{\xi_j}. \tag{9}$$

We have also

$$(x^a, \dot{x}^b) \circ \varepsilon_r = (x^a, \rho_k^b(x)y^k), \qquad (x^a, \xi_i) \circ \varepsilon_c = (x^a, \sigma_i^b(x)p_b).$$

**Remark.** In [7] by *pseudo-Lie algebroids* (resp. *pre-Lie algebroids*), we called the linear (resp. linear and skew-symmetric) Leibniz tensors on $E^*$. Throughout this note the corresponding algebroid structures (represented by pairs $(E, \varepsilon)$ or, equivalently, by $(E^*, \Lambda_\varepsilon)$) will be called simply *algebroids* (*skew algebroids*). Note also that the notion of a skew algebroid was introduced by Kosmann-Schwarzbach and Magri in [11] under the name of pre-Lie algebroid. The relation to the canonical definition of Lie algebroid is given by the following theorem.

**Theorem 1** [7]. *An algebroid structure $(E, \varepsilon)$ can be equivalently defined as a bilinear bracket $[\cdot, \cdot]_\varepsilon$ on sections of $\tau : E \to M$ together with vector bundle morphisms $a_{\mathrm{l}}^\varepsilon, a_{\mathrm{r}}^\varepsilon : E \to \mathsf{T}M$ (left and right anchors) such that*

$$[fX, gY]_\varepsilon = f(a_{\mathrm{l}}^\varepsilon \circ X)(g)Y - g(a_{\mathrm{r}}^\varepsilon \circ Y)(f)X + fg[X, Y]_\varepsilon \tag{10}$$

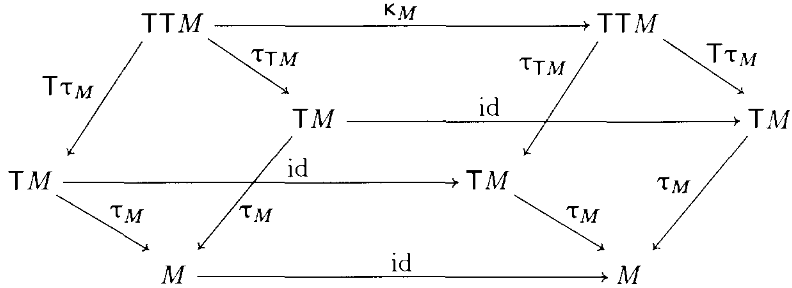
*for $f, g \in C^\infty(M)$, $X, Y \in \otimes^1(\tau)$.*

*The bracket and anchors are related to the Leibniz tensor $\Lambda_\varepsilon$ by the formulae*

$$\begin{aligned} \iota([X,Y]_\varepsilon) &= \{\iota(X),\iota(Y)\}_{\Lambda_\varepsilon}, \\ \pi^*(a^\varepsilon_{\mathrm{l}}\circ X(f)) &= \{\iota(X),\pi^* f\}_{\Lambda_\varepsilon}, \\ \pi^*(a^\varepsilon_{\mathrm{r}}\circ X(f)) &= \{\pi^* f,\iota(X)\}_{\Lambda_\varepsilon}. \end{aligned} \tag{11}$$

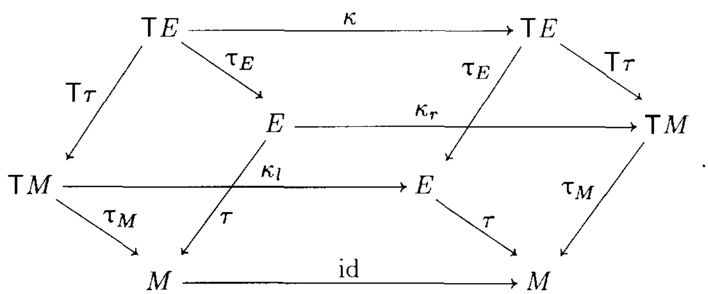
*We have also $a^\varepsilon_{\mathrm{l}} = \varepsilon_{\mathrm{r}}$ and $a^\varepsilon_{\mathrm{r}} = (\varepsilon_c)^*$. The algebroid $(E,\varepsilon)$ is a Lie algebroid if and only if the tensor $\Lambda_\varepsilon$ is a Poisson tensor.*

The canonical example of a mapping $\varepsilon$ in the case of $E = \mathsf{T}M$ is given by $\varepsilon = \varepsilon_M = \alpha_M^{-1}$ – the inverse to the Tulczyjew isomorphism, which can be defined as the dual to the isomorphism of double vector bundles




. (12)

In general, the Leibniz structure map $\varepsilon$ is not an isomorphism and, consequently, its dual $\kappa^{-1} = \varepsilon^{*_r}$ with respect to the right projection




. (13)

is a relation and not a mapping. In this diagram $\kappa_r = \varepsilon_r$.

Similarly as in [5], we can extend $\varepsilon$ to mappings

$$\varepsilon^{\otimes r} : \otimes^r_E \mathsf{T}^* E \longrightarrow \mathsf{T}\otimes^r_M E^*,$$

$r \geqslant 0$, as follows. We put, for $r = 0$,

$$\begin{aligned} \varepsilon^{\otimes 0} &: E\times\mathbb{R}\to\mathsf{T}(M\times\mathbb{R}) = \mathsf{T}M\times\mathbb{R}\times\mathsf{T}_0\mathbb{R} = \mathsf{T}M\times\mathbb{R}\times\mathbb{R} \\ &: (e,t)\mapsto(\varepsilon_r(e),t,0) \end{aligned} \tag{14}$$

and, for $r = 1$, $\varepsilon^{\otimes 1} = \varepsilon$. For $r > 1$, we apply the tangent functor to the tensor product map

$$\otimes^r : E^*\times_M\cdots\times_M E^* \longrightarrow \otimes^r_M E^*$$

and we define $\varepsilon^{\otimes r}$ by the commutativity of the diagram

$$\begin{array}{ccc} \mathsf{T}E^* \times_{\mathsf{T}M} \cdots \times_{\mathsf{T}M} \mathsf{T}E^* & \xrightarrow{\mathsf{T}\otimes^r} & \mathsf{T}\otimes^r_M E^* \\ \uparrow{\scriptstyle \times^r \varepsilon} & & \uparrow{\scriptstyle \varepsilon^{\otimes r}} \\ \mathsf{T}^*E \times_E \cdots \times_E \mathsf{T}^*E & \xrightarrow{\otimes^r} & \otimes^r_E \mathsf{T}^*E \end{array} . \tag{15}$$

In local coordinates, we have

$$(x^a, \xi_{i_1\cdots i_r}, \dot{x}^b, \dot{\xi}_{j_1\cdots j_r}) \circ \varepsilon^{\otimes r} = \left(x^a, \pi_{i_1\cdots i_r}, \rho^b_k y^k, \sum_l (c^k_{ij_l} y^i \pi_{j_1\cdots j_{l-1}kj_{l+1}\cdots j_r} + \sigma^{a'}_{j_l} \pi_{j_1\cdots j_{l-1}a'j_{l+1}\cdots j_r})\right). \tag{16}$$

Of course, $\varepsilon^{\otimes r}$ may be reduced to skew-symmetric (symmetric) tensors and, as in [5], we have also the commutative diagram

$$\begin{array}{ccc} (\otimes^i_E \mathsf{T}^*E) \times_E (\otimes^{r-i}_E \mathsf{T}^*E) & \xrightarrow{\varepsilon^{\otimes i} \times \varepsilon^{\otimes(r-i)}} & (\mathsf{T}\otimes^i_M E^*) \times_{\mathsf{T}M} (\mathsf{T}\otimes^{r-i}_M E^*) \\ \downarrow{\scriptstyle \otimes_E} & & \downarrow{\scriptstyle \mathsf{T}\otimes_M} \\ \otimes^r_E \mathsf{T}^*E & \xrightarrow{\varepsilon^{\otimes r}} & \mathsf{T}\otimes^r_M E^* \end{array} . \tag{17}$$

## 2. Leibniz relations

In the following definition we adapt standard concepts of the theory of Poisson manifolds and Lie algebroids.

**Definition 2.** A submanifold $N$ of a Leibniz manifold $(M, \Lambda)$ is called *coisotropic* if $\widetilde{\Lambda}((\mathsf{T}N)^0) \subset \mathsf{T}N$, where $(\mathsf{T}N)^0 \subset \mathsf{T}^*_N M$ is the annihilator of $\mathsf{T}N \subset \mathsf{T}M$. A relation $L \subset M_1 \times M_2$ between Leibniz manifolds $(M_1, \Lambda_1)$ and $(M_2, \Lambda_2)$ is a *Leibniz relation* if $L$ is a coisotropic submanifold of Leibniz manifold $(M_1 \times M_2, (-\Lambda_1) \times \Lambda_2)$.

**Theorem 2.** *Let $F : M_1 \to M_2$ be a differentiable mapping between Leibniz manifolds $(M_1, \Lambda_1)$ and $(M_2, \Lambda_2)$. Then, $\Lambda_1$ and $\Lambda_2$ are $F$-related if and only if the graph of $F$ is a Leibniz relation.*

**Proof.** The proof is completely parallel to that in the Poisson case. □

**Definition 3.** Let $\tau_i : E_i \to M_i$ be a vector bundle, $i = 1, 2$. Let $\varepsilon_i$ be an algebroid structure on $\tau_i$ and let $\Lambda_i$ be the corresponding Leibniz structure on $E_i^*$. A vector bundle morphism $\psi : E_1 \to E_2$ is an *algebroid morphism* if the dual relation $\psi^* : E_2^* \to E_1^*$ is a Leibniz relation.

## 3. The algebroid lift $\mathrm{d}_{\mathsf{T}}^{\varepsilon}$

For a tensor field $K \in \otimes^k(\tau)$, we can define the vertical lift $\mathrm{v}_\tau(K) \in \otimes^k(\tau_E)$ (cf. [6,22]). In local coordinates,

$$\mathrm{v}_\tau(f^{i_1\cdots k} e_{i_1} \otimes \cdots \otimes e_{i_k}) = f^{i_1\cdots k} \partial_{y^{i_1}} \otimes \cdots \otimes \partial_{y^{i_k}}. \tag{18}$$

A particular case of the vertical lift is the lift $\mathrm{v}_{\mathsf{T}}(K)$ of a contravariant tensor field $K$ on $M$ into a contravariant tensor field on $\mathsf{T}M$.

**Lemma 1.** *Let $(E, \varepsilon)$ be an algebroid and let $K \in \otimes^k(\tau)$, $k \geqslant 0$. Then*

$$\iota(\mathrm{v}_\tau(K)) = \mathrm{v}_{\mathsf{T}}(\iota(K)) \circ \varepsilon^{\otimes k}. \tag{19}$$

**Proof.** In local coordinates, both sides of (19) are equal

$$f^{i_1\cdots i_k}(x)\pi_{i_1\cdots i_k}$$

for $k > 0$. For a function $f$ on $M$, we get

$$\iota(\mathrm{v}_\tau f)(e, t) = tf(\tau(e)) = tf(\tau_M(\varepsilon_r(e))) = \mathrm{v}_{\mathsf{T}}(\iota(f)) \circ \varepsilon^0(e, t). \qquad \square$$

It is well known (see [5,22]) that in the case of $E = \mathsf{T}M$ we have also the tangent lift $\mathrm{d}_{\mathsf{T}} : \otimes(\tau_M) \to \otimes(\tau_{\mathsf{T}M})$ which is a $\mathrm{v}_{\mathsf{T}}$-derivation. It turns out that the presence of such a lift for a vector bundle is equivalent to the presence of an algebroid structure.

**Theorem 3.** *Let $(E, \varepsilon)$ be an algebroid. For $K \in \otimes^k(\tau)$, $k \geqslant 0$, the equality*

$$\iota(\mathrm{d}_{\mathsf{T}}^{\varepsilon}(K)) = \mathrm{d}_{\mathsf{T}}(\iota(K)) \circ \varepsilon^{\otimes k} \tag{20}$$

*defines the tensor field $\mathrm{d}_{\mathsf{T}}^{\varepsilon}(K) \in \otimes^k(\tau_E)$ which is linear and the mapping*

$$\mathrm{d}_{\mathsf{T}}^{\varepsilon} : \otimes(\tau) \longrightarrow \otimes(\tau_E)$$

*is a $\mathrm{v}_\tau$-derivation of degree 0. In local coordinates,*

$$\mathrm{d}_{\mathsf{T}}^{\varepsilon}(f^i(x)e_i) = f_i(x)\sigma_i^a(x)\partial_{x^a} + \left(y^i\rho_i^a(x)\frac{\partial f^k}{\partial x^a}(x) + c_{ij}^k(x)y^i f^j(x)\right)\partial_{y^k}. \tag{21}$$

*Conversely, if $D : \otimes(\tau) \to \otimes(\tau_E)$ is a $\mathrm{v}_\tau$-derivation of degree 0 such that $D(K)$ is linear for each $K \in \otimes^1(\tau)$, then there is an algebroid structure $\varepsilon$ on $\tau : E \to M$ such that $D = \mathrm{d}_{\mathsf{T}}^{\varepsilon}$.*

**Proof.** Since $\mathrm{d}_{\mathsf{T}}(\iota(K)) \circ \varepsilon^{\otimes k}$ is a linear function on $\otimes_E^k \mathsf{T}^*E$, it defines a unique tensor field $\mathrm{d}_{\mathsf{T}}^{\varepsilon}(K) \in \otimes^k(\tau_E)$. In local coordinates,

$$\begin{aligned}\mathrm{d}_{\mathsf{T}}(\iota(f^{i_1\cdots i_k}e_{i_1} \otimes \cdots \otimes e_{i_k})) &= \mathrm{d}_{\mathsf{T}}(f^{i_1\cdots i_k}\xi_{i_1\cdots i_k}) \\ &= \frac{\partial f^{i_1\cdots i_k}}{\partial x^a}\dot{x}^a\xi_{i_1\cdots i_k} + f^{i_1\cdots i_k}\dot{\xi}_{i_1\cdots i_k}.\end{aligned} \tag{22}$$

Hence,

$$\mathrm{d}_{\mathsf{T}}(\iota(f^{i_1\cdots i_k}e_{i_1}\otimes\cdots\otimes e_{i_k}))\circ\varepsilon^{\otimes k}$$
$$=\frac{\partial f^{i_1\cdots i_k}}{\partial x^a}\rho_i^a y^i\pi_{i_1\cdots i_k}+f^{i_1\cdots i_k}\sum_l(c^r_{ij_l}y^i\pi_{j_1\cdots j_{l-1}rj_{l+1}\cdots j_r}+\sigma^{a'}_{j_l}\pi_{j_1\cdots j_{l-1}a'j_{l+1}\cdots j_r}) \tag{23}$$

and

$$\mathrm{d}^{\varepsilon}_{\mathsf{T}}(f^{i_1\cdots i_k}e_{i_1}\otimes\cdots\otimes e_{i_k})=\frac{\partial f^{i_1\cdots i_k}}{\partial x^a}\rho_i^a y^i\partial_{y^{i_1}}\otimes\cdots\otimes\partial_{y^{i_k}}$$
$$+f^{i_1\cdots i_k}\sum_l\partial_{y^{i_1}}\otimes\cdots\otimes(c^j_{ii_l}y^i\partial_{y^j}+\sigma^a_{i_l}\partial_{x^a})\otimes\cdots\otimes\partial_{y^{i_k}}. \tag{24}$$

Now, it is easy to see that $\mathrm{d}^{\varepsilon}_{\mathsf{T}}(K)$ is linear and that $\mathrm{d}^{\varepsilon}_{\mathsf{T}}$ is a $\mathrm{v}_\tau$-derivation of order 0.

We can give another, more intrinsic, proof of the fact that $\mathrm{d}^{\varepsilon}_{\mathsf{T}}$ is a $\mathrm{v}_\tau$-derivation.

Let $K\in\otimes^k(\tau)$, $L\in\otimes^l(\tau)$ and let us consider $\widetilde{K}=\iota(K)\circ\otimes^k$, $\widetilde{L}=\iota(L)\circ\otimes^l$, and $\widetilde{K\otimes L}=\widetilde{K}\cdot\widetilde{L}$ as functions on $(\times^{k+l}_M E^*)\times_M(\times^l_M)$. Since

$$\mathrm{d}_{\mathsf{T}}(\widetilde{K}\widetilde{L})=\mathrm{d}_{\mathsf{T}}\widetilde{K}\mathrm{v}_{\mathsf{T}}\widetilde{L}+\mathrm{v}_{\mathsf{T}}\widetilde{K}\,\mathrm{d}_{\mathsf{T}}\widetilde{L}$$

(cf. [5]), we get

$$\mathrm{d}_{\mathsf{T}}(\widetilde{K}\widetilde{L})\circ(\times^{k+l}\varepsilon)=(\mathrm{d}_{\mathsf{T}}\widetilde{K}\circ(\times^k\varepsilon))(\mathrm{v}_{\mathsf{T}}\widetilde{L}\circ(\times^l\varepsilon))+(\mathrm{v}_{\mathsf{T}}\widetilde{K}\circ(\times^k\varepsilon))(\mathrm{d}_{\mathsf{T}}\widetilde{L}\circ(\times^l\varepsilon))$$

and, in view of Lemma 1,

$$\mathrm{d}^{\varepsilon}_{\mathsf{T}}(K\otimes L)=\mathrm{d}^{\varepsilon}_{\mathsf{T}}(K)\otimes\mathrm{v}_\tau(L)+\mathrm{v}_\tau(K)\otimes\mathrm{d}^{\varepsilon}_{\mathsf{T}}(L).$$

To prove the converse, we use the method similar to the method used in the proof that derivations of $\mathcal{C}^\infty(M)$ are given by vector fields.

Let $D:\otimes\tau\to\otimes(\tau_E)$ be a $\mathrm{v}_\tau$-derivation of degree zero. It follows that $D:\mathcal{C}^\infty(M)\to\mathcal{C}^\infty(E)$ is also a $\mathrm{v}_\tau$-derivation. Consequently,

$$D(1)=D(1\cdot 1)=D(1)\mathrm{v}_\tau(1)+\mathrm{v}_\tau(1)D(1)=2D(1) \tag{25}$$

and $D(1)=0$.

It is well known that for every $m_0\in M$ we can find local coordinates $(x^a)$ in a neighborhood $U$ of $m_0$, $x^a(m_0)=0$, given by globally defined functions, such that for each $f\in\mathcal{C}^\infty(M)$ we have

$$f(m)=f(m_0)+(x^a f_a)(m)$$

for $m\in U$ and some $f_a\in\mathcal{C}^\infty(M)$. It is clear that $f_a(m_0)=(\partial f/\partial x^a)(m_0)$.

Let $e\in E$, $\tau(e)=m_0$. We have

$$(Df)(e)=D(f(m_0))(e)+D(x^a)(e)\mathrm{v}_\tau(f_a)(e)+\mathrm{v}_\tau(x^a)(e)D(f_a)(e)$$
$$=f_a(m_0)D(x^a)(e)=\frac{\partial f}{\partial x^a}(m_0)D(x^a)(e). \tag{26}$$

We can apply this argument to every point $m \in U$ and coordinates $x^a - x^a(m)$ and, since $D(x^a - x^a(m)) = D(x^a)$, we obtain the local formula

$$D(f) = \frac{\partial f}{\partial x^a} D(x^a). \tag{27}$$

Let $D(K)$ be linear for $K \in \otimes^k(\tau)$. Then the functions $D(x^a)$ are linear on fibers of $E$ and, consequently, we can write $D(x^a) = \rho^a_j y^j$ for some functions $\rho^a_j \in C^\infty(U)$, and

$$D(f) = \frac{\partial f}{\partial x^a} \rho^a_j y^j.$$

Let $(e_i)$ be a basis of local sections of $E$. We have, due to linearity of $D(e_i)$,

$$D(e_i) = \sigma^a_i \partial_{x^a} + c^k_{ji} y^j \partial_{y^k}, \tag{28}$$

where $\sigma^a_i, c^k_{ji} \in C^\infty(U)$. Finally, the equality

$$\begin{aligned} D(f^i e_i) &= D(f^i)\mathrm{v}_\tau(e_i) + \mathrm{v}_\tau(f^i) D(e_i) \\ &= \frac{\partial f^i}{\partial x^a} \rho^a_j y^j \partial_{y^i} + f^i c^k_{ji} y^j \partial_{y^k} + f^i \sigma^a_i \partial_{x^a} \end{aligned} \tag{29}$$

shows that, when restricted to functions and sections of $E$, the derivation $D$ equals $\mathrm{d}^\varepsilon_\mathrm{T}$ for $\varepsilon$ as in (8) when acting on functions and sections of $E$. But the $\mathrm{v}_\tau$-derivation $D$ of order 0 is uniquely determined by its values on functions and on sections of $E$. □

**Theorem 4.** *For $X, Y \in \otimes^1(\tau)$,*

$$[\mathrm{v}_\tau(X), \mathrm{d}^\varepsilon_\mathrm{T}(Y)] = \mathrm{v}_\tau([X, Y]_\varepsilon). \tag{30}$$

**Proof.** Easy calculations in local coordinates. □

**Corollary 1.** *For $X \in \otimes^1(\tau)$ and $Y \in \otimes^k(\tau)$, we have*

$$[\mathrm{v}_\tau(X), \mathrm{d}^\varepsilon_\mathrm{T}(Y)] = \mathrm{v}_\tau([X, Y]_\varepsilon)$$

*and*

$$[\mathrm{v}_\tau(Y), \mathrm{d}^\varepsilon_\mathrm{T}(X)] = \mathrm{v}_\tau([Y, X]_\varepsilon),$$

*where*

$$[X, Y_1 \otimes \cdots \otimes Y_k] = \sum_i Y_1 \otimes \cdots \otimes [X, Y_i] \otimes \cdots \otimes Y_k,$$

$$[Y_1 \otimes \cdots \otimes Y_k, X] = \sum_i Y_1 \otimes \cdots \otimes [Y_i, X] \otimes \cdots \otimes Y_k,$$

*and the formulae for $[\ ,\ ]_\varepsilon$ are similar.*

**Proof.** The proof follows directly from Theorem 4 and from the fact that $\mathrm{d}^\varepsilon_\mathrm{T}$ is a $\mathrm{v}_\tau$-derivation of order 0. □

The algebroid lift $\mathrm{d}_{\mathsf{T}}^{\varepsilon}$ may be used to an alternative definition of the bracket $[\ ,\ ]_{\varepsilon}$ (cf. [5] for the case of the canonical Lie algebroid in the tangent bundle).

Let $X$ be a section of $E$. At points of $X(M) \subset E$, we have the decomposition of $\mathsf{T}E$ into the horizontal (tangent to $X(M)$) and the vertical part. For a vector field $V$ on $E$, we denote by $X^*V$ the unique section of $E$ such that the vertical lift $\mathrm{v}_\tau(X^*V)$ is, on $X(M)$, the vertical part of $V$.

**Theorem 5.** *Let $(E, \varepsilon)$ be an algebroid and let $X, Y \in \otimes^1(\tau)$. Then*

$$X^*(\mathrm{d}_{\mathsf{T}}^{\varepsilon} Y) = [X, Y]_{\varepsilon}. \tag{31}$$

**Proof.** We have to prove that the vector field $\mathrm{d}_{\mathsf{T}}^{\varepsilon}(Y) - \mathrm{v}_\tau([X, Y]_\varepsilon)$ is tangent to $X(M) \subset E$. Let $X = f^i e_i$ and $Y = g^i e_i$, in local coordinates, and let $\varepsilon$ be as in (8). We have, according to (10), (18), and (21),

$$\begin{aligned} \mathrm{d}_{\mathsf{T}}^{\varepsilon}(g^j e_j) - \mathrm{v}_\tau([f^i e_i, g^j e_j]_\varepsilon) &= c^k_{ij} y^i g^j \partial_{y_k} + \rho^a_i y^i \frac{\partial g^k}{\partial x^a} \partial_{y^k} + \sigma^a_i g^i \partial_{x^a} \\ &\quad - c^k_{ij} f^i g^j \partial_{y_k} - \rho^a_i f^i \frac{\partial g^k}{\partial x^a} \partial_{y^k} + \sigma^a_i g^i \frac{\partial f^k}{\partial x^a} \partial_{y^k} \\ &= \sigma^a_i g^i \left( \partial_{x^a} + \frac{\partial f^k}{\partial x^a} \partial_{y^k} \right), \end{aligned} \tag{32}$$

since $y^i = f_i$ on $X(M)$. It is clear that the vector fields $\partial_{x^a} + (\partial f^k / \partial x^a) \partial_{y^k}$ are tangent to $X(M)$. □

Let $\varphi : \mathsf{T}^*E \to \mathsf{T}E^*$ be a morphism of vector bundles over $E^*$. The vector bundle dual to $\tau_{E^*} : \mathsf{T}E^* \to E^*$ can be identified (via $\mathcal{R}_\tau$) with the vector bundle $\mathsf{T}^*\tau : \mathsf{T}^*E \to E^*$ and vice versa. It follows that the dual to $\varphi$ can be identified with a morphism

$$\varphi^+ : \mathsf{T}^*E \to \mathsf{T}E^*. \tag{33}$$

**Theorem 6.** *Let $\varepsilon$ be an algebroid structure as in (8). Then the morphism*

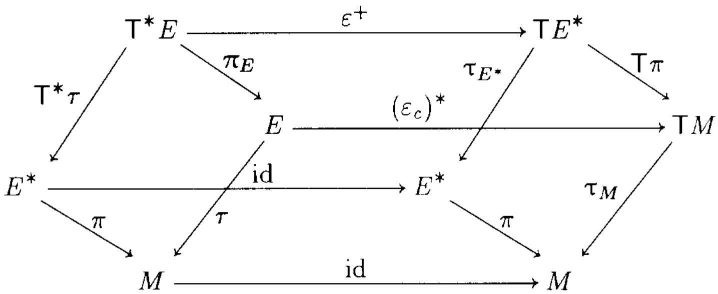
$$\varepsilon^+ : \mathsf{T}^*E \longrightarrow \mathsf{T}E^*,$$

*dual to $\varepsilon$ with respect to the vector bundle structures over $E^*$ is again an algebroid structure on $E$. The corresponding Leibniz tensor $\Lambda_{\varepsilon^+}$ is the transposition of the tensor $\Lambda_\varepsilon$. In local coordinates,*

$$(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) \circ \varepsilon^+ = (x^a, \pi_i, \sigma^b_k(x) y^k, c^k_{ji}(x) y^i \pi_k + \rho^a_j(x) p_a). \tag{34}$$

**Proof.** The proof is an immediate consequence of the general properties of a dual to a double vector bundle. We refer to [8,20] for the theory of duality in the category of double vector bundles. □

It follows from the general theory of double vector bundles that $(\varepsilon^+)_r = (\varepsilon_c)^*$ and $(\varepsilon^+)_c = (\varepsilon_r)^*$. Thus, we have the diagram




(35)

We call an algebroid structure $(E, \varepsilon)$ *skew-symmetric* if $\varepsilon = -\varepsilon^+$, where the multiplication by $-1$ is with respect to the vector bundle structure over $E^*$. It is clearly equivalent to the fact that $\Lambda_\varepsilon$ is skew-symmetric. Skew-symmetric algebroids are sometimes called *pre-Lie algebroids* [7,11].

## 4. The tangent and cotangent lifts of an algebroid

The theory of the tangent and cotangent lifts of algebroids is perfectly analogous to that of Lie algebroids (see [2,6]). If $\varepsilon$ is an algebroid structure on $\tau : E \to M$ and $\Lambda_\varepsilon$ is the corresponding Leibniz tensor on $E^*$, the tangent lift $\mathrm{d_T}(\Lambda_\varepsilon)$ is linear with respect to both vector bundle structures on $\mathsf{T}E^*$ and, consequently, it defines algebroid structures on dual bundles: $\mathsf{T}^*E^* \to E^*$ and $\mathsf{T}E \to \mathsf{T}M$, called respectively the *cotangent* and *tangent* lift of the algebroid $\varepsilon$. Let us denote the corresponding double vector bundle morphisms by $\mathcal{T}^*\varepsilon$ and $\mathcal{T}\varepsilon$, respectively. It is easy to prove the following theorem.

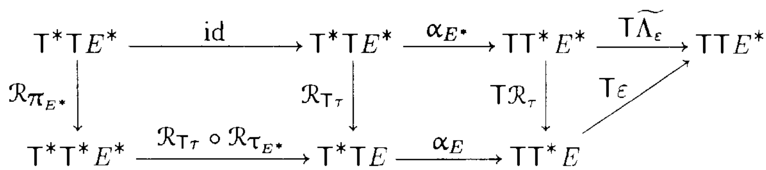
**Theorem 7.** *The tangent $\mathcal{T}\varepsilon$ and the cotangent $\mathcal{T}^*\varepsilon$ lifts of an algebroid structure $\varepsilon$ on $\tau : E \to M$ are defined by the following commutative diagram*

$$\begin{array}{ccccc}
\mathsf{TT}^*E & \xrightarrow{\alpha_E} & \mathsf{T}^*\mathsf{T}E & \xrightarrow{\mathcal{R}\pi_{E^*} \circ \mathcal{R}_{\mathsf{T}\pi}} & \mathsf{T}^*\mathsf{T}^*E^* \\
{\scriptstyle \mathsf{T}\varepsilon}\downarrow & & {\scriptstyle \mathcal{T}\varepsilon}\downarrow & & {\scriptstyle \mathcal{T}^*\varepsilon}\downarrow \\
\mathsf{TT}E^* & \xrightarrow{\kappa_{E^*}} & \mathsf{TT}E^* & \xrightarrow{\mathrm{id}} & \mathsf{TT}E^*
\end{array} \quad . \tag{36}$$

**Proof.** We have two commutative diagrams

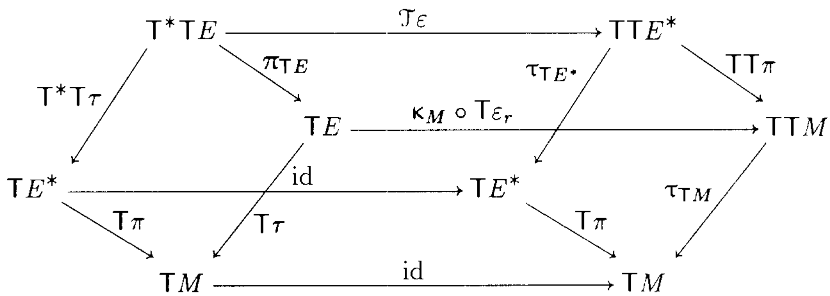
$$\begin{array}{ccc}
\mathsf{T}^*E^* & \xrightarrow{\widetilde{\Lambda_\varepsilon}} & \mathsf{T}E^* \\
{\scriptstyle \mathcal{R}_\tau}\downarrow & \nearrow{\scriptstyle \varepsilon} & \\
\mathsf{T}^*E & &
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathsf{T}^*\mathsf{T}E^* & \xrightarrow{\alpha_{E^*}} & \mathsf{TT}^*E^* \\
{\scriptstyle \mathcal{R}_{\mathsf{T}\tau}}\downarrow & & {\scriptstyle \mathsf{T}\mathcal{R}_\tau}\downarrow \\
\mathsf{T}^*\mathsf{T}E & \xrightarrow{\alpha_E} & \mathsf{TT}^*E
\end{array} \quad . \tag{37}$$

It follows that the diagram



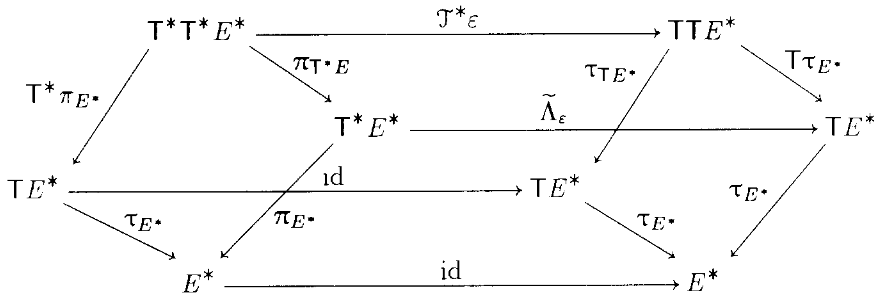

(38)

is also commutative. All the mappings in this diagram respect three vector bundle structures and, since $\widetilde{\mathrm{d_T}\Lambda_\varepsilon} = \kappa_{E^*} \circ \mathsf{T}\widetilde{\Lambda}_\varepsilon \circ \alpha_{E^*}$ (cf. [5]), we have $\mathcal{T}\varepsilon$, $\mathcal{T}^*\varepsilon$ as in the diagram (36), and the corresponding double vector bundle morphisms:




(39)

and




(40)

We see from these diagrams, that $(\mathcal{T}\varepsilon)_r = \kappa_M \circ \mathsf{T}\varepsilon_r$ and $(\mathcal{T}^*\varepsilon)_r = \widetilde{\Lambda}_\varepsilon$. We have also

$$(\mathcal{T}\varepsilon)_c = \mathsf{T}\varepsilon_c \circ \alpha_M^{-1}, \qquad (\mathcal{T}^*\varepsilon)_c = \widetilde{\Lambda}_\varepsilon \qquad \square \tag{41}$$

**Theorem 8.** *Let $\varepsilon$ be an algebroid structure on $\tau : E \to M$. The following properties of $\varepsilon$ are equivalent*:

(a) *$\varepsilon$ is a Lie algebroid structure,*

(b) *$\Lambda_\varepsilon$ is a Poisson tensor,*

(c) *$\Lambda_E$ and $\mathrm{d_T}\Lambda_\varepsilon$ are $\varepsilon$-related,*

(d) *The following diagram is commutative (on the domain of the relation $\mathsf{T}^*\varepsilon$)*:

$$
\begin{array}{ccccc}
\mathsf{TT}^*E & \xleftarrow{\ \widetilde{\Lambda}_E\ } & \mathsf{T}^*\mathsf{T}^*E & & \\
\Big\downarrow{\scriptstyle \mathsf{T}\varepsilon} & & \Big\uparrow{\scriptstyle \mathsf{T}^*\varepsilon} & & \\
\mathsf{TT}E^* & & \mathsf{T}^*\mathsf{T}E^* & & \\
\Big\uparrow{\scriptstyle \kappa_{E^*}} & & & \searrow{\scriptstyle \alpha_{E^*}^{-1}} & \\
\mathsf{TT}E^* & \xleftarrow{\ \mathsf{T}\varepsilon\ } & \mathsf{TT}^*E & \xleftarrow{\ \mathsf{T}\mathcal{R}_\tau\ } & \mathsf{TT}^*E^*
\end{array}, \tag{42}
$$

(e) $\mathrm{d}^\varepsilon_{\mathsf{T}}([X,Y]_\varepsilon) = [\mathrm{d}^\varepsilon_{\mathsf{T}}(X), \mathrm{d}^\varepsilon_{\mathsf{T}}(Y)]$ *for all* $X, Y \in \otimes^1(\tau)$.

**Proof.** (a)$\Longleftrightarrow$(b) by the definition of a Lie algebroid.

(b)$\Longleftrightarrow$(c)$\Longleftrightarrow$(d) is a version of Theorems 4.4 and 4.5 from [5]: a Leibniz tensor $\Lambda$ is a Poisson tensor on a manifold $N$ if and only if $\Lambda_N$ and $-\mathrm{d}_{\mathsf{T}}\Lambda$ are $\widetilde{\Lambda}$-related.

Finally, (c)$\Longleftrightarrow$(e), since

$$\iota_{\mathsf{T}^*E}([\mathrm{d}^\varepsilon_{\mathsf{T}}(X), \mathrm{d}^\varepsilon_{\mathsf{T}}(Y)]) = \{\iota\,\mathrm{d}^\varepsilon_{\mathsf{T}}(X), \iota\,\mathrm{d}^\varepsilon_{\mathsf{T}}(Y)\}_{\Lambda_E} = \{\mathrm{d}_{\mathsf{T}}(\iota(X))\circ\varepsilon, \mathrm{d}_{\mathsf{T}}(\iota(Y))\circ\varepsilon\}_{\Lambda_E},$$

which equals

$$\begin{aligned}
&\{\mathrm{d}_{\mathsf{T}}(\iota(X)), \mathrm{d}_{\mathsf{T}}(\iota(Y))\}_{\mathrm{d}_{\mathsf{T}}\Lambda_\varepsilon}\circ\varepsilon \\
&\quad = \mathrm{d}_{\mathsf{T}}(\{\iota(X), \iota(Y)\}_{\Lambda_\varepsilon})\circ\varepsilon\,\mathrm{d}_{\mathsf{T}}(\iota([X,Y]_\varepsilon))\circ\varepsilon = \iota(\mathrm{d}^\varepsilon_{\mathsf{T}}([X,Y]_\varepsilon))
\end{aligned}$$

if and only if $\Lambda_E$ and $\mathrm{d}_{\mathsf{T}}\Lambda_\varepsilon$ are $\varepsilon$-related. □

## 5. The Lie and exterior derivatives

Let $(E,\varepsilon)$ be an algebroid, $X \in \otimes^1(\tau)$, $\mu \in \otimes^1(\pi)$. Since $\mathrm{d}^\varepsilon_{\mathsf{T}}(X)$ is a linear vector field on $E$, $\mathrm{d}^\varepsilon_{\mathsf{T}}(\iota(\mu))$ is a linear function on $E$. The corresponding section of $E^*$ we call the *Lie derivative of* $\mu$ *along* $X$ and denote $\mathcal{L}^\varepsilon_X(\mu)$, i.e.,

$$\iota(\mathcal{L}^\varepsilon_X(\mu)) = \mathrm{d}^\varepsilon_{\mathsf{T}}(X)(\iota(\mu)). \tag{A}$$

In local coordinates,

$$\mathcal{L}^\varepsilon_{f^i e_i}(\mu_j e^j_*) = f^i \sigma^b_i \frac{\partial \mu_k}{\partial x^b} e^k_* + \left(f^i c^k_{ji} + \frac{\partial f^k}{\partial x^a}\rho^a_j\right)\mu_k e^j_*. \tag{B}$$

It is easy to see that

$$\mathcal{L}^\varepsilon_X(f\mu) = \mathcal{L}^\varepsilon_X(\mu) + a^\varepsilon_{\mathrm{r}}(X)(f)\cdot\mu \tag{C}$$

and

$$\mathcal{L}^\varepsilon_{fX}(\mu) = f\mathcal{L}^\varepsilon_X(\mu) + \langle X,\mu\rangle\,\mathrm{d}^\varepsilon_{\mathrm{l}}(f), \tag{D}$$

where *the left derivative* $\mathrm{d}_{\mathrm{l}}^{\varepsilon}(f) \in \otimes^1(\pi)$ is defined by

$$\langle \mathrm{d}_{\mathrm{l}}^{\varepsilon}(f), Y\rangle = \langle \mathrm{d}f, a_{\mathrm{l}}^{\varepsilon}(Y)\rangle, \quad Y \in \otimes^1(\tau).$$

Similarly, we can define the *right derivative*

$$\langle \mathrm{d}_{\mathrm{r}}^{\varepsilon}(f), Y\rangle = \langle \mathrm{d}f, a_{\mathrm{r}}^{\varepsilon}(Y)\rangle, \quad Y \in \otimes^1(\tau).$$

We have clearly

$$\mathrm{d}_{\mathrm{r}}^{\varepsilon}(fg) = f\,\mathrm{d}_{\mathrm{r}}^{\varepsilon}(g) + g\,\mathrm{d}_{r}^{\varepsilon}(f) \tag{E}$$

and the analog identity for the left derivative.

**Theorem 9.** *For every $X \in \otimes^1(\tau)$ there is a unique derivative $\mathcal{L}_X^{\varepsilon}$ of the tensor algebra $\mathfrak{T}(\tau)$, called the Lie derivative along $X$, such that*
(a) $\mathcal{L}_X^{\varepsilon}(f) = a_{\mathrm{r}}^{\varepsilon}(X)(f) = \langle X, \mathrm{d}_{\mathrm{r}}^{\varepsilon} f\rangle$ *for* $f \in \mathcal{C}^{\infty}(M)$,
(b) $\iota(\mathcal{L}_X^{\varepsilon}(\mu)) = \mathrm{d}_{\mathrm{T}}^{\varepsilon}(X)(\iota(\mu))$ *for* $\mu \in \otimes^1(\pi)$,
(c) $\mathcal{L}_X^{\varepsilon}(Y) = -[Y, X]_{\varepsilon}$ *for* $Y \in \otimes^1(\tau)$.
*Moreover,*
(d) $\mathcal{L}_{fX}^{\varepsilon}(K) = f\mathcal{L}_X^{\varepsilon}(K) + \mathrm{i}_{\mathrm{d}_{\mathrm{l}}^{\varepsilon}(f)\otimes X}(K)$ *for* $f \in C^{\infty}(M)$ *and* $K \in \mathfrak{T}(\tau)$.

**Proof.** Any derivative of the tensor algebra $\mathfrak{T}(\tau)$ is uniquely determined by its values on functions and sections of $E$ and $E^*$. On the other hand, it is easy to see that the identities

$$\mathcal{L}_X^{\varepsilon}(f \cdot g) = \mathcal{L}_X^{\varepsilon}(f) \cdot g + f \cdot \mathcal{L}_X^{\varepsilon}(g), \qquad \mathcal{L}_X^{\varepsilon}(f\mu) = \mathcal{L}_X^{\varepsilon}(f)\mu + f\mathcal{L}_X^{\varepsilon}(\mu),$$

and (43)

$$\mathcal{L}_X^{\varepsilon}(fY) = \mathcal{L}_X^{\varepsilon}(f)Y + f\mathcal{L}_X^{\varepsilon}(Y),$$

for $f, g \in \mathcal{C}^{\infty}(M)$, $\mu \in \otimes^1(\pi)$, $Y \in \otimes^1(\tau)$, which follow from (C), (E), and the properties of $[\ ,\ ]_{\varepsilon}$, are sufficient to extend $\mathcal{L}_X^{\varepsilon}$ to a derivation of the tensor algebra

$$\mathcal{L}_X^{\varepsilon}(A_1 \otimes \cdots \otimes A_k) = \sum_i A_1 \otimes \cdots \otimes \mathcal{L}_X^{\varepsilon}(A_i) \otimes \cdots \times A_k.$$

It remains to prove (d). Since $\mathrm{i}_{\mathrm{d}_{\mathrm{l}}^{\varepsilon}(f)\otimes X}$ is also a derivation, it is sufficient to check (d) on functions and sections of $E$ and $E^*$. On functions $\mathrm{i}_{\mathrm{d}_{\mathrm{l}}^{\varepsilon}(f)\otimes X}$ acts trivially and (d) follows from (a). For $\mu \subset \otimes^1(\mu)$, we have $\mathrm{i}_{\mathrm{d}_{\mathrm{l}}^{\varepsilon}(f)\otimes X}\mu = \langle X, \mu\rangle\,\mathrm{d}_{\mathrm{l}}^{\lambda}(f)$ and (d) follows from (D). Finally,

$$\mathrm{i}_{\mathrm{d}_{\mathrm{l}}^{\varepsilon}(f)\otimes X}(Y) = -a_{\mathrm{l}}^{\varepsilon}(Y)(f) \cdot X = -[Y, fX]_{\varepsilon} + f[Y, X]_{\varepsilon}$$

for $Y \in \otimes^1(\tau)$ and (d) follows from (c).

**Remark.** (A) was introduced in [16] as the definition of $\mathrm{d}_{\mathrm{T}}^{\varepsilon}(X)$ for $\varepsilon$ being a Lie algebroid structure.

**Theorem 10.** *Let* $\otimes^1(\tau) \ni X \to \mathcal{L}_X \in \mathrm{Der}(\otimes(\pi))$ *be a linear mapping assigning to each section of* $E$ *a derivative of the tensor algebra* $\otimes(\pi)$. *Then* $\mathcal{L}_X = \mathcal{L}^\varepsilon_X$ *for an algebroid structure* $\varepsilon$ *on* $E$ *if and only if* $\mathcal{L}_{fX} = f\mathcal{L}_X + \mathrm{i}_{\mathrm{d_l}(f)\otimes X}$ *for a derivation* $\mathrm{d_l} : C^\infty(M) \to \otimes^1(\pi)$.

**Proof.** The 'only if' part follows from the previous theorem. To prove the 'if' part, we can start with the identity

$$\iota(\mathcal{L}_X\mu) = \mathrm{D}(X)(\iota(\mu))$$

to define a lift $\mathrm{D}(X)$ of $X$ to a tangent vector field on $E$, and to show that this lift is of the form $\mathrm{d}^\varepsilon_{\mathsf{T}}$.

We can also find directly the bracket $[\ ,\ ]_\varepsilon$ as follows. First, since

$$(\mathrm{i}_X\mathcal{L}_Y - \mathcal{L}_Y\mathrm{i}_X)(f\mu) = f\,(\mathrm{i}_X\mathcal{L}_Y - \mathcal{L}_Y\mathrm{i}_X)\,(\mu),$$

there is an element $[X, Y] \in \otimes^1(\tau)$ such that $\mathrm{i}_X\mathcal{L}_Y - \mathcal{L}_Y\mathrm{i}_X = \iota_{[X,Y]}$. The bracket $[\,,\,]$ is bilinear and

$$(\mathrm{i}_{fX}\mathcal{L}_{gY} - \mathcal{L}_{gY}\mathrm{i}_{fX}) = fg(\mathrm{i}_X\mathcal{L}_Y\mathcal{L}_Y\mathrm{i}_X) + f\mathrm{i}_X\mathrm{i}_{\mathrm{d_l}g\otimes Y} - g\mathcal{L}_Y(f)\cdot\iota_X,$$

which shows that $[\ ,\ ]$ is an algebroid bracket with the left anchor defined by $a_\mathrm{l}(X)(g) = \langle X, \mathrm{d_l}g\rangle$ and the right anchor defined by $a_\mathrm{r}(Y)(f) = \mathcal{L}_Y(f)$. The right anchor is really tensorial with respect to $Y$, since

$$\mathcal{L}_{gY}(f) = g\mathcal{L}_Y(f) + \mathrm{i}_{\mathrm{d_l}g\otimes Y}f = g\mathcal{L}_Y(f).$$

Hence, $[\ ,\ ] = [\ ,\ ]_\varepsilon$ for an algebroid structure $\varepsilon$ and it is easy to see that $\mathcal{L}_X = \mathcal{L}^\varepsilon_X$. □

We have already defined the right and the left exterior derivatives

$$\mathrm{d}^\varepsilon_\mathrm{r},\ \mathrm{d}^\varepsilon_\mathrm{l} : C^\infty(M) \to \otimes^1(\pi)$$

with the property

$$\mathcal{L}^\varepsilon_X(f) = \mathrm{i}_X\,\mathrm{d}^\varepsilon_\mathrm{r} f + \mathrm{d}^\varepsilon_\mathrm{l}\mathrm{i}_X f, \quad f \in C^\infty(M). \tag{44}$$

In general, for $\varepsilon$ which is not skew-symmetric, it is not true that $\mathcal{L}^\varepsilon_X = \mathrm{i}_X\,\mathrm{d}^\varepsilon + \mathrm{d}^\varepsilon\mathrm{i}_X$ for some exterior derivative $\mathrm{d}^\varepsilon$ even on the Grassman algebra $\Phi(\pi)$. However, we can always find

$$\mathrm{d}^\varepsilon_\mathrm{r},\ \mathrm{d}^\varepsilon_\mathrm{l} : \otimes^1(\pi) \to \otimes^2(\pi)$$

such that

$$\mathcal{L}^\varepsilon_X\mu - \mathrm{d}^\varepsilon_\mathrm{l}\mathrm{i}_X\mu = \mathrm{i}^1_X\,\mathrm{d}^\varepsilon_\mathrm{r}\mu = -\mathrm{i}^2_X\,\mathrm{d}^\varepsilon_\mathrm{l}\mu.$$

We have, in local coordinates,

$$\begin{aligned} \mathrm{d}^\varepsilon_\mathrm{r}(f_k e^k_*) &= c^k_{ij} f_k e^j_* \otimes e^i_* + \sigma^a_i \frac{\partial f_k}{\partial x^a} e^i_* \otimes e^k_* - \sigma^a_i \frac{\partial f_k}{\partial x^a} e^k_* \otimes e^i_*, \\ \mathrm{d}^\varepsilon_\mathrm{l}(f_k e^k_*) &= -c^k_{ij} f_k e^i_* \otimes e^j_* - \sigma^a_i \frac{\partial f_k}{\partial x^a} e^k_* \otimes e^i_* + \sigma^a_i \frac{\partial f_k}{\partial x^a} e^i_* \otimes e^k_*. \end{aligned} \tag{45}$$

It follows also that

$$\begin{aligned} \mathrm{d}_\mathrm{r}^\varepsilon(f\mu) &= f\,\mathrm{d}_\mathrm{r}^\varepsilon(\mu) + \mathrm{d}_\mathrm{r}^\varepsilon(f)\otimes\mu - \mu\otimes\mathrm{d}_\mathrm{l}^\varepsilon(f),\\ \mathrm{d}_\mathrm{l}^\varepsilon(f\mu) &= f\,\mathrm{d}_\mathrm{l}^\varepsilon(\mu) + \mathrm{d}_\mathrm{l}^\varepsilon(f)\otimes\mu - \mu\otimes\mathrm{d}_\mathrm{r}^\varepsilon(f), \end{aligned} \tag{46}$$

and $\mathrm{d}_\mathrm{l}^{\varepsilon^+} = \mathrm{d}_\mathrm{r}^\varepsilon$.

The following theorem is the general algebroid version of Theorem 15 (d) of [6].

**Theorem 11.** *There is a unique derivation* $\mathrm{d}_\mathrm{l}^\varepsilon : \otimes^1(\pi) \to \otimes^2(\pi)$ *such that*

$$\mathcal{L}_X^\varepsilon\mu = \mathrm{d}_\mathrm{l}^\varepsilon \mathrm{i}_X\mu - \mathrm{i}_X^2\,\mathrm{d}_\mathrm{l}^\varepsilon\mu, \quad \mu\in\otimes^1(\pi),\ X\in\otimes^1(\tau).$$

*Moreover,*

$$\mathrm{v}_\pi(\mathrm{d}_\mathrm{l}^\varepsilon\mu) = -\mathcal{L}_{\mathrm{v}_\pi(\mu)}\Lambda^\varepsilon.$$

**Proof.** We have

$$\begin{aligned} \mathrm{v}_\pi(\mathrm{d}_\mathrm{l}^\varepsilon(f_k e_*^k)) &= -c_{ij}^k f_k \partial_{\xi_i}\otimes\partial_{\xi_j} - \sigma_i^a\frac{\partial f_k}{\partial x^a}\partial_{\xi_k}\otimes\partial_{\xi_i} + \rho_i^a\frac{\partial f_k}{\partial x^a}\partial_{\xi_i}\otimes\partial_{\xi_k}\\ &= -\mathcal{L}_{f_k\partial_{\xi_k}}(c_{ij}^k\xi_k\partial_{\xi_i}\otimes\partial_{\xi_j} + \rho_i^a\partial_{\xi_i}\otimes\partial_{x^a} - \sigma_i^a\partial_{x^a}\otimes\partial_{\xi_j})\\ &= -\mathcal{L}_{\mathrm{v}_\pi(\mu)}\Lambda^\varepsilon. \qquad \square \end{aligned}$$

**Remark.** It is well known that for skew-symmetric $\varepsilon$ we have $\mathrm{d}_\mathrm{r}^\varepsilon = \mathrm{d}_\mathrm{l}^\varepsilon = \mathrm{d}^\varepsilon$ and, consequently, it can be extended to a derivation $\mathrm{d}^\varepsilon$ of the Grassman algebra $\Phi(\pi)$ [11]. Then, $\mathrm{d}^\varepsilon\circ\mathrm{d}^\varepsilon = 0$ if and only if $\Lambda^\varepsilon$ is Poisson, i.e., $\varepsilon$ is a Lie algebroid structure.

## 6. Bialgebroids

Let $\varepsilon$ and $\widetilde{\varepsilon}$ be algebroid structures on bundles $\tau : E \to M$ and $\pi : E^* \to M$ respectively. Following the ideas of Xu and Mackenzie [15, Theorem 6.2], we call the pair $(\varepsilon, \widetilde{\varepsilon})$ a bialgebroid if $\varepsilon$, regarded as a vector bundle morphism

$$\begin{array}{ccc} \mathrm{T}^*E & \xrightarrow{\ \varepsilon\ } & \mathrm{T}E^* \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \mathrm{T}\pi} \\ E & \xrightarrow{\ \varepsilon_\tau\ } & \mathrm{T}M \end{array}, \tag{47}$$

is a morphism of algebroids $\mathcal{T}^*\widetilde{\varepsilon}$ and $\mathcal{T}\widetilde{\varepsilon}$, the cotangent and the tangent lifts of the algebroid $\widetilde{\varepsilon}$. This means exactly that the dual relation $\varepsilon^*$ between the corresponding dual bundles

$$\begin{array}{ccc} \mathrm{T}E & \xleftarrow{\ \varepsilon^*\ } & \mathrm{T}E \\ {\scriptstyle \tau_E}\downarrow & & \downarrow{\scriptstyle \mathrm{T}\tau} \\ E & \xrightarrow{\ \tau\ } & \mathrm{T}M \end{array}; \tag{48}$$

is a Leibniz relation between the Leibniz tensors $\mathrm{d_T}\Lambda_{\tilde{\varepsilon}}$ on both sides, i.e., this relation is a coisotropic submanifold of $\mathsf{T}E \times \mathsf{T}E$ with respect to the Leibniz tensor $\mathrm{d_T}\Lambda_{\tilde{\varepsilon}} \times (-\mathrm{d_T}\Lambda_{\tilde{\varepsilon}})$. Let us assume that, in local coordinates, $\varepsilon$ has the form (8) and

$$\Lambda_{\tilde{\varepsilon}} = \widetilde{c}_k^{ij} y^k \partial_{y^i} \otimes \partial_{y^j} + \widetilde{\rho}^{ia} \partial_{\xi_i} \otimes \partial_{x^b} - \widetilde{\sigma}^{aj} \partial_{x^a} \otimes \partial_{y^j}. \tag{49}$$

It is easy to calculate that

$$\begin{aligned} \mathrm{d_T}(\Lambda_{\tilde{\varepsilon}}) = {} & \left( \frac{\partial \widetilde{c}_k^{ij}}{\partial x^a} y^k \dot{x}^a + \widetilde{c}_k^{ij} y^k \dot{y}^k \right) \partial_{\dot{y}^i} \otimes \partial_{\dot{y}^j} + \frac{\partial \widetilde{\rho}^{ib}}{\partial x^a} \dot{x}^a \partial_{\dot{y}^i} \otimes \partial_{\dot{x}^b} \\ & - \frac{\partial \widetilde{\sigma}^{aj}}{\partial x^b} \dot{x}^b \partial_{\dot{x}^a} \otimes \partial_{\dot{y}^j} + \widetilde{c}_k^{ij} y^k (\partial_{\dot{y}^i} \otimes \partial_{y^j} + \partial_{y^i} \otimes \partial_{\dot{y}^j}) \\ & + \widetilde{\rho}^{ib} (\partial_{\dot{y}^i} \otimes \partial_{x^b} + \partial_{y^i} \otimes \partial_{\dot{x}^b}) - \widetilde{\sigma}^{aj} (\partial_{\dot{x}^a} \otimes \partial_{y^j} + \partial_{x^a} \otimes \partial_{\dot{y}^j}) \end{aligned} \tag{50}$$

and that the relation $\varepsilon^*$ in $\mathsf{T}E \times \mathsf{T}E$ with coordinates $(x^a, y^i, \dot{x}^b, \dot{y}^j, \bar{x}^{a'}, \bar{y}^{i'}, \dot{\bar{x}}^{b'}, \dot{\bar{y}}^{j'})$ is defined by equations

$$\begin{aligned} & F_0^a = \bar{x}^a - x^a = 0, && F_1^a = \dot{x}^a - \sigma_k^b(x) \bar{y}^k = 0, \\ & F_2^j = \dot{y}^j - c_{ik}^j(x) y^i \bar{y}^k - \dot{\bar{y}}^j, && F_3^a = \dot{\bar{x}}^a - \rho_k^a(x) y^k = 0. \end{aligned} \tag{51}$$

The equations define a coisotropic submanifold $N$ with respect to $\Lambda = (-\mathrm{d_T}\Lambda_{\tilde{\varepsilon}}) \times \mathrm{d_T}\Lambda_{\tilde{\varepsilon}}$ if and only if the Leibniz bracket of the functions (51) vanish on $N$. The Hamiltonian vector fields corresponding to these functions are the following (we make use of the fact that $x^a$ and $\bar{x}^a$ are equal on $N$):

$$\begin{aligned} X_0^a = {} & \widetilde{\sigma}^{aj} (\partial_{\dot{y}^j} + \partial_{\dot{\bar{y}}^j}), \\ X_1^b = {} & \left( \widetilde{\sigma}^{aj} \frac{\partial \sigma_k^b}{\partial x^a} \bar{y}^k - \frac{\partial \widetilde{\sigma}^{bj}}{\partial x^a} \sigma_1^a \bar{y}^l \right) \partial_{\dot{y}^j} - \widetilde{\sigma}^{bj} \partial_{y^j} + \sigma_i^b \widetilde{c}_k^{ij} \bar{y}^k \partial_{\dot{\bar{y}}^j} + \sigma_i^b \widetilde{\rho}^{ia} \partial_{\dot{\bar{x}}^a}, \\ X_3^b = {} & \left( \widetilde{\sigma}^{aj} \frac{\partial \rho_k^b}{\partial x^a} y^k - \rho_i^b \widetilde{c}_k^{ij} y^k \right) \partial_{\dot{y}^j} + \widetilde{\sigma}^{bj} \partial_{\bar{y}^j} + \frac{\partial \widetilde{\sigma}^{bj}}{\partial x^a} \rho_1^a {}^{\backprime} y^l \partial_{\dot{\bar{y}}^j} - \rho_i^b \widetilde{\rho}^{ia} \partial_{\dot{\bar{x}}^a}, \\ X_2^j = {} & \left( \frac{\partial \widetilde{c}_k^{jl}}{\partial x^a} \sigma_i^a y^k \bar{y}^i + \widetilde{c}_k^{jl} c_{si}^k y^s \bar{y}^i + \widetilde{c}_k^{jl} \dot{\bar{y}}^k + \widetilde{\sigma}^{al} \frac{\partial c_{ik}^j}{\partial x^a} y^i \bar{y}^k - c_{ik}^j \widetilde{c}_s^{il} y^s \bar{y}^k \right) \partial_{\dot{y}^l} \\ & + \left( \frac{\partial \widetilde{c}_k^{jl}}{\partial x^a} \rho_i^a y^i \bar{y}^k + \widetilde{c}_k^{jl} \dot{\bar{y}}^k + c_{ik}^j \widetilde{c}_s^{kl} y^i \bar{y}^s \right) \partial_{\dot{\bar{y}}^l} + \left( \frac{\partial \widetilde{\rho}^{jb}}{\partial x^a} \sigma_k^a \bar{y}^k - \widetilde{\rho}^{ib} c_{ik}^j \bar{y}^k \right) \partial_{\dot{x}^b} \\ & + \left( \frac{\partial \widetilde{\rho}^{jb}}{\partial x^a} \rho_k^a y^k + \widetilde{\rho}^{kb} c_{ik}^j \bar{y}^i \right) \partial_{\dot{\bar{x}}^b} + \widetilde{c}_k^{ji} y^k \partial_{y^i} + \widetilde{\rho}^{jb} \partial_{x^b} + \widetilde{c}_k^{ji} \bar{y}^k \partial_{\bar{y}^i} + \widetilde{\rho}^{jb} \partial_{\bar{x}^b}. \end{aligned} \tag{52}$$

One can easily see that $F_0^a$ commute, with respect to $\{\ ,\ \}_\Lambda$, with all defining functions, so that we have three types of functions left. Moreover, $X_1^b(F_1^a) = 0$, $X_3^b(F_3^a) = 0$ and we get the following seven non-trivial equations defining bialgebroids, corresponding, respectively, to $X_1^b(F_3^a)=0$, $X_3^b(F_1^a)=0$, $X_3^b(F_2^j)=0$, $X_2^j(F_1^b)=0$, $X_1^b(F_2^j)=0$, $X_2^j(F_3^b)=0$, and $X_2^j(F_2^i)=0$.

**Theorem 12.** *The tensors $\Lambda_\varepsilon$ and $\Lambda_{\tilde\varepsilon}$ as in (9) and (49) constitute a bialgebroid structure if and only if the following equations are satisfied:*

$$(1)\qquad \sigma_i^b\tilde\rho^{ia}+\tilde\sigma^{bi}\rho_i^a=0,$$

$$(2)\qquad \rho_i^b\tilde\rho^{ia}+\tilde\sigma^{bi}\sigma_i^a=0,$$

$$(3)\qquad \frac{\partial\rho_k^b}{\partial x^a}\tilde\sigma^{aj}-\frac{\partial\tilde\sigma^{bj}}{\partial x^a}\rho_k^a-\tilde c_k^{ij}\rho_i^b-c_{ki}^j\tilde\sigma^{bi}=0,$$

$$(4)\qquad -\frac{\partial\sigma_j^b}{\partial x^a}\tilde\rho^{ka}+\frac{\partial\tilde\rho^{kb}}{\partial x^a}\sigma_j^a-\tilde c_j^{ki}\sigma_i^b-c_{ij}^k\tilde\rho^{ka}=0,$$

$$(5)\qquad \frac{\partial\sigma_k^b}{\partial x^a}\tilde\sigma^{aj}-\frac{\partial\tilde\sigma^{bj}}{\partial x^a}\sigma_k^a-\tilde\sigma_k^{ij}\sigma_i^b+c_{ik}^j\tilde\sigma^{bi}=0,$$

$$(6)\qquad -\frac{\partial\rho_k^b}{\partial x^a}\tilde\rho^{ja}+\frac{\partial\tilde\rho^{jb}}{\partial x^a}\rho_k^a-\tilde\sigma_k^{ij}\rho_i^b-c_{ki}^j\tilde\rho^{ib}=0,$$

$$(7)\qquad \frac{\partial\tilde c_i^{jl}}{\partial x^a}\sigma_k^a+\frac{\partial c_{ik}^j}{\partial x^a}\tilde\sigma^{al}-\frac{\partial\tilde c_k^{jl}}{\partial x^a}\rho_i^a-\frac{\partial c_{ik}^l}{\partial x^a}\tilde\rho^{ja}+\tilde c_s^{jl}c_{ik}^s-\tilde c_i^{sl}c_{sk}^j$$
$$-\tilde c_k^{sl}c_{is}^j-\tilde c_i^{js}c_{sk}^l-\tilde c_k^{js}c_{is}^l=0.$$

**Corollary 2.** *The pair $(\varepsilon,\tilde\varepsilon)$ constitutes a bialgebroid if and only if $(\tilde\varepsilon,\varepsilon)$ constitutes a bialgebroid.*

**Proof.** The family of equations (1)–(7) does not change when we interchange 'tilde' with 'no tilde'. □

The original definition of a Lie bialgebroid by Mackenzie and Xu [15] was given in terms cf exterior derivatives and Lie brackets. In the case of a general algebroid we have a slight substitute of the exterior derivative only, but it is enough to get the full analogy.

**Theorem 13.** *A pair $(\varepsilon,\tilde\varepsilon)$ of algebroid structures on $E$ and $E^*$, respectively, constitutes a bialgebroid if and only if*

$$\mathrm{d}_1^{\tilde\varepsilon}[X,Y]_\varepsilon=[\mathrm{d}_1^{\tilde\varepsilon}X,Y]_\varepsilon+[X,\mathrm{d}_1^{\tilde\varepsilon}Y]_\varepsilon \tag{53}$$

*for all $X,Y\in\otimes^1(\tau)$, where the brackets ('the Schouten brackets') are defined by the formulae*

$$\begin{aligned}[][X\otimes Y,Z]_\varepsilon&=[X,Z]_\varepsilon\otimes Y+X\otimes[Y,Z]_\varepsilon,\\ [X,Y\otimes Z]_\varepsilon&=[Z,X]_\varepsilon\otimes Y+X\otimes[Z,Y]_\varepsilon.\end{aligned} \tag{54}$$

**Proof.** We shall show, in local coordinates, that (53) is equivalent to Eqs. (1)–(7). To reduce the problem to the case $X = e_i$, $Y = e_j$, we find the relation of (53) with the structure of $\mathcal{C}^\infty(M)$-module in $\otimes^1(\tau)$. Replacing in (53) $Y$ by $fY$ we get

$$\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}([X,f]_\varepsilon Y + f[X,Y]_\varepsilon) = [\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}X, f]_\varepsilon^2 \otimes Y + Y \otimes [\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}X, f]_\varepsilon^1 + f[\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}X, Y]_\varepsilon \\ +[X, f\,\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}Y + \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}(f)\otimes Y - Y \otimes \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}(f)]_\varepsilon. \tag{55}$$

(We use the following conventions: $[X,f]_\varepsilon = a_\mathrm{l}^\varepsilon(X)(f)$, $[f,X] = -a_\mathrm{r}^\varepsilon(X)(f)$, $[X\otimes Y, f]_\varepsilon^2 = [Y,f]_\varepsilon X$, $[X\otimes Y, f]_\varepsilon^1 = [X,f]_\varepsilon Y$, etc.). We get furtherly

$$\begin{aligned}&[X,f]_\varepsilon \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}Y + \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}([X,f]_\varepsilon)\otimes Y - Y\otimes \mathrm{d}_\mathrm{r}^{\tilde\varepsilon}[X,f]_\varepsilon + f\,\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}[X,Y]_\varepsilon \\ &\quad + \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}(f)\otimes[X,Y]_\varepsilon - [X,Y]_\varepsilon \otimes \mathrm{d}_\mathrm{r}^{\tilde\varepsilon}(f) \\ &\qquad = [X,f]_\varepsilon \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}Y + f[X, \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}Y]_\varepsilon + [X, \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}(f)]_\varepsilon \otimes Y + \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}(f)\otimes[X,Y]_\varepsilon \\ &\qquad\quad - [X,Y]_\varepsilon \otimes \mathrm{d}_\mathrm{r}^{\tilde\varepsilon}(f) - Y\otimes[X, \mathrm{d}_\mathrm{r}^{\tilde\varepsilon}(f)]_\varepsilon \\ &\qquad\quad + [\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}X, f]_\varepsilon^2 \otimes Y + Y\otimes[\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}X, f]_\varepsilon^1 + f[\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}X, Y]_\varepsilon\end{aligned} \tag{56}$$

and, finally,

$$\begin{aligned}&f(\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}[X,Y]_\varepsilon - [\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}X, Y]_\varepsilon - [X, \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}Y]_\varepsilon) \\ &\quad = Y\otimes(\mathrm{d}_\mathrm{r}^{\tilde\varepsilon}[X,f]_\varepsilon + [\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}X, f]_\varepsilon^1 - [X, \mathrm{d}_\mathrm{r}^{\tilde\varepsilon}(f)]_\varepsilon) \\ &\qquad - (\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}[X,f]_\varepsilon - [\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}X, f]_\varepsilon^2 - [X, \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}(f)]_\varepsilon)\otimes Y.\end{aligned} \tag{57}$$

This shows that (53) with $Y = e_i$ is equivalent to equations

$$\mathrm{d}_\mathrm{r}^{\tilde\varepsilon}[X,f]_\varepsilon + [\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}X, f]_\varepsilon^1 - [X, \mathrm{d}_\mathrm{r}^{\tilde\varepsilon}(f)]_\varepsilon = 0, \tag{a}$$

$$\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}[X,f]_\varepsilon - [\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}X, f]_\varepsilon^2 - [X, \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}(f)]_\varepsilon = 0, \tag{b}$$

where $X \in \otimes^1(\tau)$ and $f \in \mathcal{C}^\infty(M)$.

We reduce (a) and (b) once more, this time with respect to $X$. Let us put $gX$ instead of $X$. We get from (a)

$$\mathrm{d}_\mathrm{r}^{\tilde\varepsilon}(g[X,f]_\varepsilon) + [g\,\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}X + \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}g\otimes X - X\otimes \mathrm{d}_\mathrm{r}^{\tilde\varepsilon}g, f]_\varepsilon^1 \\ -g[X, \mathrm{d}_\mathrm{r}^{\tilde\varepsilon}(f)]_\varepsilon - [g, \mathrm{d}_\mathrm{r}^{\tilde\varepsilon}(f)]_\varepsilon X = 0. \tag{58}$$

Hence,

$$g(\mathrm{d}_\mathrm{r}^{\tilde\varepsilon}[X,f]_\varepsilon + [\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}X, f]_\varepsilon^1 - [X, \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}(f)]_\varepsilon) + ([\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}g, f]_\varepsilon - [g, \mathrm{d}_\mathrm{r}^{\tilde\varepsilon}f]_\varepsilon)X = 0,$$

which shows that (a) is equivalent to

$$[\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}(g), f]_\varepsilon - [g, \mathrm{d}_\mathrm{r}^{\tilde\varepsilon}(f)]_\varepsilon = 0, \tag{c}$$

$$\mathrm{d}_\mathrm{r}^{\tilde\varepsilon}[e_i, f]_\varepsilon + [\mathrm{d}_\mathrm{l}^{\tilde\varepsilon}e_i, f]_\varepsilon^1 - [e_i, \mathrm{d}_\mathrm{r}^{\tilde\varepsilon}(f)]_\varepsilon = 0. \tag{d}$$

Similarly, (b) is equivalent to

$$[\mathrm{d}_\mathrm{r}^{\tilde\varepsilon}(g), f]_\varepsilon - [g, \mathrm{d}_\mathrm{l}^{\tilde\varepsilon}]_\varepsilon = 0, \tag{c'}$$

$$\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}}[e_i, f]_{\varepsilon} + [\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}} e_i \cdot f]_{\varepsilon}^2 - [e_i, \mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}}(f)]_{\varepsilon} = 0. \tag{d'}$$

Relaxing now the $C^{\infty}(M)$-module structure in (53) with respect to $X$, we get analogously

$$\mathrm{d}_{\mathrm{r}}^{\tilde{\varepsilon}}[f, Y]_{\varepsilon} + [f, \mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}} Y]_{\varepsilon}^1 + [\mathrm{d}_{\mathrm{r}}^{\tilde{\varepsilon}}(f), Y]_{\varepsilon} = 0, \tag{e}$$

$$\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}}[f, Y]_{\varepsilon} - [f, \mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}} Y]_{\varepsilon}^2 - [\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}}(f), Y]_{\varepsilon} = 0, \tag{f}$$

which are equivalent to (c), (c′) and (e), (f) with $Y = e_i$. Finally, in local basis, (53) is equivalent to the system of equations

(1′) $[\mathrm{d}_{\mathrm{r}}^{\tilde{\varepsilon}}(g), f]_{\varepsilon} - [g, \mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}}(f)]_{\varepsilon} = 0,$

(2′) $[\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}}(g), f]_{\varepsilon} - [g, \mathrm{d}_{\mathrm{r}}^{\tilde{\varepsilon}}(f)]_{\varepsilon} = 0,$

(3′) $\mathrm{d}_{\mathrm{r}}^{\tilde{\varepsilon}}[e_i, f]_{\varepsilon} + [\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}} e_i, f]_{\varepsilon}^1 - [e_i, \mathrm{d}_{\mathrm{r}}^{\tilde{\varepsilon}}(f)]_{\varepsilon} = 0,$

(4′) $\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}}[f, e_i]_{\varepsilon} - [\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}}(f), e_i]_{\varepsilon} - [f, \mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}} e_i]_{\varepsilon}^2 = 0,$

(5′) $\mathrm{d}_{\mathrm{r}}^{\tilde{\varepsilon}}[f, e_i]_{\varepsilon} + [\mathrm{d}_{\mathrm{r}}^{\tilde{\varepsilon}}(f), e_i]_{\varepsilon} + [f, \mathrm{d}_{\mathrm{r}}^{\tilde{\varepsilon}} e_i]_{\varepsilon}^1 = 0,$

(6′) $\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}}[e_i, f]_{\varepsilon} - [\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}} e_i, f]_{\varepsilon}^2 - [e_i, \mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}}(f)]_{\varepsilon} = 0,$

(7′) $\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}}[e_i, e_j]_{\varepsilon} + [\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}} e_i, e_j]_{\varepsilon} - [e_i, \mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}} e_j]_{\varepsilon} = 0.$

Now, it is a direct check that these equations are equivalent to the system of equations (1)–(7). □

A canonical example of a bialgebroid is given by the following theorem.

**Theorem 14.** *Let $\varepsilon$ be a Lie algebroid structure on $\tau : E \to M$ and let $\Lambda \in \otimes^2(\tau)$. Let $\widetilde{\varepsilon}$ be the algebroid structure on $E^*$ which corresponds to the linear Leibniz structure $\mathrm{d}_{\mathrm{T}}^{\varepsilon}\Lambda$ on $E$. Then the pair $(\varepsilon, \widetilde{\varepsilon})$ is a bialgebroid.*

**Proof.** From Theorem 11 we get

$$\mathrm{v}_{\tau}(\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}} X) = -[\mathrm{v}_{\tau}(X), \mathrm{d}_{\mathrm{T}}^{\varepsilon}\Lambda] = -\mathrm{v}_{\tau}([X, \Lambda]_{\varepsilon}),$$

where we used a formula from Corollary 1. It follows that $\mathrm{d}_{\mathrm{l}}^{\tilde{\varepsilon}} X = -[X, \Lambda]_{\varepsilon}$ and (53) reads now

$$[[X, Y]_{\varepsilon}, \Lambda] = [[X, \Lambda]_{\varepsilon}, Y]_{\varepsilon} + [X, [Y, \Lambda]_{\varepsilon}]_{\varepsilon},$$

which easily follows from the Jacobi identity for $[\ ,\ ]_{\varepsilon}$. □

**Remark.** A standard example of a situation as above is provided by the Lie bialgebroid induced by a Poisson structure on a manifold (cf. [11,15]). Moreover, the above theorem shows that a bialgebroid may be constituted by a Lie algebroid and an algebroid which is not even skew-symmetric.

**Example.** An extreme case are bialgebroids over a single point. It means exactly that we have algebra structures $[\ ,\ ]$ and $[\ ,\ ]'$ on a finite-dimensional vector space $E$ and on its dual $E^*$ respectively. (An algebra structure on $E$ is a bilinear operation $[\ ,\ ] : E\times E \to E$.) They form a bialgebroid (or, simply, bialgebra) if and only if

$$\mathrm{d}'[X, Y] = [\mathrm{d}'X, Y] + [X, \mathrm{d}'Y]$$

for all $X, Y \in E$, where $\mathrm{d}' : E \to E\otimes E$ is the dual map to $[\ ,\ ]' : E^*\otimes E^* \to E^*$. In the case of Lie algebra we recognize the definition of a Lie bialgebra.

## 7. The algebroid of a linear connection on $\mathsf{T}M$

Important examples of algebroids which are not skew-symmetric are provided by linear connections on a tangent bundle. Let a linear connection be given on $\mathsf{T}M$. It can be represented by the covariant derivative

$$(X, Y) \mapsto \nabla_X Y, \qquad X, Y \in \otimes^1(\tau_M) \tag{59}$$

or, equivalently, by the horizontal projection

$$P_{\mathrm{h}} : \mathsf{TT}M \to \mathsf{TT}M$$

or by the vertical projection

$$P_{\mathrm{v}} : \mathsf{TT}M \to \mathsf{TT}M.$$

The linearity of the connection implies that $P_{\mathrm{h}}$, $P_{\mathrm{v}}$ are double vector bundle morphisms.

Let $(x^a, \dot{x}^b, x'^c, \dot{x}'^d)$ be a coordinate system on $\mathsf{TT}M$. We have

$$\begin{aligned}(x^a, \dot{x}^b, x'^c, \dot{x}'^d)\circ P_{\mathrm{v}} &= (x^a, \dot{x}^b, 0, \dot{x}'^d + \Gamma^d_{ba}(x)\dot{x}^a x'^b),\\ (x^a, \dot{x}^b, x'^c, \dot{x}'^d)\circ P_{\mathrm{h}} &= (x^a, \dot{x}^b, x'^c, -\Gamma^d_{ba}(x)\dot{x}^a x'^b),\\ \nabla_X Y &= \frac{\partial Y^a}{\partial x^b}X^b\frac{\partial}{\partial x^a} + \Gamma^a_{cb}Y^bX^c\frac{\partial}{\partial x^a}.\end{aligned} \tag{60}$$

**Theorem 15.** *There is a unique algebroid structure $\varepsilon$ on $\mathsf{T}M$ such that*

$$\mathcal{L}^{\varepsilon}_X Y = \nabla_X Y. \tag{61}$$

*For this algebroid* $a^{\varepsilon}_{\mathrm{r}} = \mathrm{id}$, $\mathrm{a}^{\varepsilon}_{\mathrm{l}} = 0$.

*Conversely, any algebroid structure $(\mathsf{T}M, \varepsilon)$ such that $a^{\varepsilon}_r = \mathrm{id}$, $\mathrm{a}^{\varepsilon}_{\mathrm{l}} = 0$ is the algebroid of a linear connection on $\mathsf{T}M$*

**Proof.** We define a bracket $[\ ,\ ]_{\varepsilon}$ of vector fields by the formula

$$[X, Y]_{\varepsilon} = -\nabla_Y X. \tag{62}$$

It is obvious that it satisfies the condition (10) with $a^\varepsilon_{\mathrm{r}} = \mathrm{id}$ and $a^\varepsilon_{\mathrm{l}} = 0$:

$$\begin{aligned}[][fX, gY]_\varepsilon &= -\nabla_{gY}(fX) = -gf\nabla_Y X - gY(f)X \\ &= gf[X, Y]_\varepsilon - gY(f)X \\ &= gf[X, Y] - ga^\varepsilon_{\mathrm{r}}(Y)(f)X + fa^\varepsilon_{\mathrm{l}}(X)(g)Y.\end{aligned}$$

To prove the converse it is enough to notice that $a^\varepsilon_{\mathrm{r}} = \mathrm{id}$, $a^\varepsilon_{\mathrm{l}} = 0$ implies the following form of $\varepsilon : \mathsf{T}^*\mathsf{T}M \to \mathsf{T}\mathsf{T}^*M$:

$$(x^a, p_b, \dot{x}^c, \dot{p}_d) \circ \varepsilon = (x_a, \pi_b, 0, f_d - \Gamma^a_{db}\dot{x}^b\pi_a), \tag{63}$$

where $(x^a, \dot{x}^b, f_c, \pi_d)$ are coordinates in $\mathsf{T}^*\mathsf{T}M$. □

Let $P_{\mathrm{h}}$ be the horizontal projection of a linear connection. The formula

$$P^{\mathrm{t}}_{\mathrm{h}} = \kappa_M \circ P_h \circ \kappa_M \tag{64}$$

defines the horizontal lift of the *transposed linear connection* on $\mathsf{T}M$. The connection is called *symmetric* if it is equal to the transposed connection, i.e., if $P_{\mathrm{h}} = P^{\mathrm{t}}_{\mathrm{h}}$.

It is well known that for each linear connection on $\mathsf{T}M$ with the covariant derivative $\nabla$ there exists the *dual connection* on $\mathsf{T}^*M$, with the covariant derivative $\nabla^+$, such that

$$X(\langle\mu, Y\rangle) = \langle\nabla^+_X\mu, Y\rangle + \langle\mu, \nabla_X Y\rangle, \quad \mu \in \otimes^1(\pi_M),\ X, Y \in \otimes^1(\tau_M).$$

**Theorem 16.** *Let $(\mathsf{T}M, \varepsilon)$ be the algebroid structure of a linear connection with the horizontal projection $P_{\mathrm{h}}$. The following relations are satisfied:*
(a) *$\varepsilon = (P^{\mathrm{t}})^+_{\mathrm{v}} \circ \varepsilon_M$, where $\varepsilon_M = \alpha_M^{-1}$ is the canonical algebroid on $\mathsf{T}M$ and $(P^t)^+_{\mathrm{v}}$ is the vertical projection of the dual to the transposed connection,*
(b) *$\widetilde{\Lambda_\varepsilon} = (P^{\mathrm{t}})^+_{\mathrm{v}} \circ \widetilde{\Lambda_M}$, where $\Lambda_M$ is the canonical Poisson tensor on $\mathsf{T}^*M$,*
(c) *$\mathrm{d}^\varepsilon_{\mathsf{T}} = P_{\mathrm{h}} \circ \mathrm{d}_{\mathsf{T}}$, i.e., $\mathrm{d}^\varepsilon_{\mathsf{T}}$ is the horizontal lift,*
(d) *$\mathcal{L}^\varepsilon_X\mu = \nabla^+_X\mu$ for $\mu \in \otimes^1(\pi_M)$,*
(e) *$\mathrm{d}^\varepsilon_{\mathrm{r}} f = \mathrm{d}f$, $\mathrm{d}^\varepsilon_{\mathrm{l}} f = 0$ for $f \in C^\infty(M)$,*
(f) *$\mathrm{d}^\varepsilon_{\mathrm{r}}\mu = \nabla^+\mu$ for $\mu \in \otimes^1(\pi_M)$.*

**Proof.**
(a) In local coordinates,

$$\begin{aligned}(x^a, p_b, \dot{x}^c, \dot{p}_d) \circ (P^{\mathrm{t}})^+_{\mathrm{v}} &= (x^a, p_b, 0, \dot{p}_d - (\Gamma^{\mathrm{t}})^a_{bd}p_a\dot{x}^b) \\ &= (x^a, p_b, 0, \dot{p}_d - \Gamma^a_{db}p_a\dot{x}^b), \\ (x^a, p_b, \dot{x}^c, \dot{p}_d) \circ (P^{\mathrm{t}})^+_{\mathrm{v}} \circ \varepsilon_M &= (x^a, \pi_b, 0, f_d - \Gamma^a_{db}\pi_a\dot{x}^b)\end{aligned} \tag{65}$$

and the equality follows from (63).
(b) We have from (a)

$$\widetilde{\Lambda_\varepsilon} = \varepsilon \circ \mathcal{R}_{\tau_M} = (P^{\mathrm{t}})^+_{\mathrm{v}} \circ \varepsilon_M \circ \mathcal{R}_{\tau_M} = (P^{\mathrm{t}})^+_{\mathrm{v}} \circ \widetilde{\Lambda_M}. \tag{66}$$

(c) It follows from the formula (21) that

$$\begin{aligned} \mathrm{d}^{\varepsilon}_{\mathsf{T}}(X) &= X^a \partial_{x^a} - \Gamma^a_{cb} \dot{x}^b X^c \partial_{\dot{x}^a} = P_{\mathrm{h}}(X^a \partial_{x^a}) \\ &= P_{\mathrm{h}}\left( X^a \partial_{x^a} + \frac{\partial X^a}{\partial x^b} \dot{x}^b \partial_{\dot{x}^a} \right) = P_{\mathrm{h}} \circ \mathrm{d}_{\mathsf{T}} X. \end{aligned} \tag{67}$$

(d) By the definition of the Lie derivative,

$$\iota(\mathcal{L}^{\varepsilon}_X(\mu)) = \mathrm{d}^{\varepsilon}_{\mathsf{T}} X(\iota(\mu)) = (P_{\mathrm{h}} \circ \mathrm{d}_{\mathsf{T}} X)(\iota(\mu)). \tag{68}$$

Since $\iota(\mu) = \mu_a \dot{x}^a$ and $P_{\mathrm{h}} \circ \mathrm{d}_{\mathsf{T}} X = X^a \partial_{x^a} - \Gamma^a_{cb} \dot{x}^b X^c \partial_{x^a}$, we get

$$(P_{\mathrm{h}} \circ \mathrm{d}_{\mathsf{T}} X)(\iota(\mu)) = \frac{\partial \mu_b}{\partial x^a} X^a \dot{x}^b - \Gamma^a_{cb} \dot{x}^b X^c \mu_a \tag{69}$$

and, consequently,

$$\mathcal{L}^{\varepsilon}_X(\mu) = \frac{\partial \mu_b}{\partial x^a} X^a \, \mathrm{d}x^b - \Gamma^a_{cb} X^c \mu_a \, \mathrm{d}^b = \nabla^+_X \mu.$$

(e)
$$\begin{aligned} \langle \mathrm{d}^{\varepsilon}_{\mathrm{r}}(f), X \rangle &= a^{\varepsilon}_{\mathrm{r}}(X)(f) = X(f) = \langle \mathrm{d}f, x \rangle, \\ \langle \mathrm{d}^{\varepsilon}_{\mathrm{l}}(f), X \rangle &= a^{\varepsilon}_{\mathrm{l}}(X)(f) = 0. \end{aligned} \tag{70}$$

(f) It follows from the definition of the exterior derivatives and from (d) that

$$\mathrm{i}^{\mathrm{l}}_X \, \mathrm{d}^{\varepsilon}_{\mathrm{r}} = \mathcal{L}^{\varepsilon}_X \mu - \mathrm{d}^{\varepsilon}_{\mathrm{l}} \mathrm{i}_X \mu = \mathcal{L}^{\varepsilon}_X \mu = \nabla^+_X \mu = \mathrm{i}^{\mathrm{l}}_X \nabla^+ \mu. \qquad \square$$

## 8. Metric connections

In this section we give an interpretation of the Levi-Civita connection in terms of algebroids. Let $g$ be a contravariant metric tensor on $M$ and let $\widetilde{g} : \mathsf{T}^*M \to \mathsf{T}M$ be the corresponding isomorphism of vector bundles. The tensor $\mathrm{d}_{\mathsf{T}} g$ on $\mathsf{T}M$ is linear and defines an algebroid $(\mathsf{T}^*M, \varepsilon)$, i.e., $\Lambda_\varepsilon = \mathrm{d}_{\mathsf{T}} g$. In local coordinates,

$$\begin{aligned} g &= g^{ab} \partial_{x^a} \otimes \partial_{x^b}, \\ \mathrm{d}_{\mathsf{T}} g &= \Lambda_\varepsilon = \frac{\partial g^{ab}}{\partial x^c} \dot{x}^c \partial_{\dot{x}^a} \otimes \partial_{\dot{x}^b} + g^{ab}(\partial_{x^a} \otimes \partial_{\dot{x}^b} + \partial_{\dot{x}^a} \otimes \partial_{x^b}), \\ [\mu, \nu]_\varepsilon &= \frac{\partial g^{ab}}{\partial x^c} \mu_a \nu_b \, \mathrm{d}x^c + g^{ab} \left( \frac{\partial \mu_c}{\partial x^a} \nu_b + \mu_a \frac{\partial \nu_c}{\partial x^b} \right) \mathrm{d}x^c. \end{aligned} \tag{71}$$

It follows that $a^{\varepsilon}_{\mathrm{l}} = \widetilde{g}$ and $a^{\varepsilon}_{\mathrm{r}} = -\widetilde{g}$.

Since $\widetilde{g}$ is an isomorphism, $\widetilde{g}^{-1}_* \Lambda_\varepsilon$ is a linear Leibniz structure on $\mathsf{T}^*M$. It defines an algebroid on $\mathsf{T}M$ with the bracket

$$[X, Y]_{\varepsilon(g)} = \widetilde{g}([\widetilde{g}^{-1} X, \widetilde{g}^{-1} Y]). \tag{72}$$

The left anchor is $a^{\varepsilon(g)}_{\mathrm{l}} = \mathrm{id}$, the right anchor is $a^{\varepsilon(g)}_{\mathrm{r}} = -\mathrm{id}$, and coordinates of the algebroid bracket are given by the formulae

$$([X, Y]_{\varepsilon(g)})^a = \frac{\partial X^a}{\partial x^b} Y^b + \frac{\partial Y^a}{\partial x^b} X^b + g^{ab} \left( \frac{\partial g_{bc}}{\partial x^d} + \frac{\partial g_{bd}}{\partial x^c} - \frac{\partial g_{cd}}{\partial x^b} \right) X^c Y^d. \tag{73}$$

The formula

$$\Lambda_{\varepsilon_g} = \tfrac{1}{2}(\Lambda_M - \widetilde{g}_*^{-1}\,\mathrm{d_T} g) \tag{74}$$

defines then an algebroid structure $(\mathsf{T}, \varepsilon_g)$ with the anchors $a_r^{\varepsilon_g} = \mathrm{id}, a_l^{\varepsilon_g} = 0$, i.e., $(\mathsf{T}M, \varepsilon_g)$ is the algebroid of a linear connection on $\mathsf{T}M$. The covariant derivative of this connection is given by the formula

$$\begin{aligned}\nabla_X Y &= -[Y, X]_{\varepsilon_g} = \tfrac{1}{2}([X, Y] + [Y, X]_{\varepsilon(g)})\\ &= \frac{\partial Y^a}{\partial x^b} X^b \partial_{x^a} + \frac{1}{2} g^{ab}\left(\frac{\partial g_{bc}}{\partial x^d} + \frac{\partial g_{bd}}{\partial x^c} - \frac{\partial g_{cd}}{\partial x^b}\right) X^c Y^d \partial_{x^a}.\end{aligned} \tag{75}$$

We recognize $\nabla_X Y$ as a covariant derivative with respect to the Levi-Civita connection of the metric $g$.

## 9. Lifting processes on differentiable groupoids

In [16] a procedure of lifting of multiplicative vector fields on Lie groupoids is described. This procedure, what the authors admit implicitly, has nothing to do with the whole groupoid structure, but rather with the structure of a fibration only. In other words, it is just the standard complete lift procedure, but applied to specific vector fields. The general scheme is based on the following theorem.

**Theorem 17.** *If $\alpha : P \to B$ is a fibration, $\gamma : B \to P$ is its section, and $X$ is a projectable vector field on $P$, tangent to $\gamma(B)$, then the complete lift $\mathrm{d_T} X$ is a vector field on $\mathsf{T}P$, tangent to the subbundle $E \subset \mathsf{T}_{\gamma(B)}P$ of $\alpha$-vertical vectors.*

**Proof.** In a neighborhood of a point $p \in \vartheta(B)$, we can choose coordinates $(x^a, f^i)$ such that $(x^a)$ are coordinates near $\alpha(p) \in B$ and $(f^i)$ are coordinates in fibers, vanishing on $\gamma(B)$.

Let us write

$$X(x, f) = g^a(x)\partial_{x^a} + h^i(x, f)\partial_{f^i},$$

where $h^i(x, 0) = 0$. Thus

$$\begin{aligned}\mathrm{d_T} X &= g^a(x)\partial_{x^a} + \frac{\partial g^a}{\partial x^b}(x)\dot{x}^b \partial_{\dot{x}^a} + h^i(x, f)\partial_{f^i}\\ &\quad + \frac{\partial h^i}{\partial x^a}(x, f)\dot{x}^a \partial_{\dot{f}^i} + \frac{\partial h^i}{\partial f^j}(x, f)\dot{f}^j \partial_{\dot{f}^i}.\end{aligned}$$

The subbundle $E \subset \mathsf{T}P$ is described by the conditions $\dot{x}^a = 0,\ f^i = 0$, so that $\mathrm{d_T} X$ is clearly tangent to $E$ and

$$\mathrm{d_T} X|_E = g^a(x)\partial_{x^a} + \frac{\partial h^i}{\partial f^j}(x, 0)\dot{f}^j \partial_{\dot{f}^i}. \qquad \square$$

What we use to lift multiplicative vector fields on a Lie groupoid $\alpha, \beta : G \leftleftarrows B$ to vector fields on the bundle $A(G)$ of the corresponding Lie algebroid is only the $\alpha$-fibration structure over the base $B$ and the fact that multiplicative vector fields have the desired properties (are star-vectors in the terminology of [16]).

On the other hand, one can use the above lift to obtain the lift $\mathrm{d}^{\varepsilon}_{\mathsf{T}}$ for the corresponding Lie algebroid. This lift for sections of $A(G)$ was introduced in [16] by the formula

$$\mathrm{d}^{\varepsilon}_{\mathsf{T}} X(\iota(\mu)) = \iota(\mathcal{L}^{\varepsilon}_{X}\mu) \tag{76}$$

(cf. Theorem 9). Since we want to present this procedure in the whole generality, we have to start with a more general object than a Lie groupoid.

A *pre-Lie groupoid* is, roughly speaking, an object we obtain by relaxing the associativity condition in the definition of a Lie groupoid. Having submersions $\alpha, \beta : G \to B$ onto the manifold $B$ of units and the inclusion map $e : B \to G$, we define the vector bundle $\tau : A(G) \to B$ as usual, to be the inverse image of the vertical bundle $V^{\alpha}G \to G$ across the embedding $e : B \to G$. We shall consider, for simplicity, $B$ to be just a submanifold of $G$, so that $A(G) = V^{\alpha}_{B}G$. Starting with a section $X : B \to A(G) \subset \mathsf{T}G$, we define the right and left prolongations of $X$ to a vector field on $G$

$$\overrightarrow{X}(g) = \mathsf{T}(R_g)X(\beta g) \quad \text{and} \quad \overleftarrow{X}(g) = \mathsf{T}(L_g)T(i)X(\alpha g), \tag{77}$$

where $R_g, L_g$ are the right and left translations, and $i : G \to G$ is the inverse mapping. Now, we define a bracket on sections of $A(G)$ putting

$$[X, Y]_{\varepsilon}(p) = [\overrightarrow{X}, \overrightarrow{Y}](p) \tag{78}$$

for $p \in B$. Let us note that $\overrightarrow{X}$ and $\overleftarrow{X}$ are no longer invariant vector fields, since the pre-groupoid product is not assumed to be associative. Hence, the bracket $[\overrightarrow{X}, \overrightarrow{Y}]$ is no longer the right-prolongation of any element of $\otimes^1(\tau)$. On the other hand, the definition (78) makes sense and we have the following.

**Theorem 18.** *The bracket* $[\ ,\ ]_{\varepsilon}$ *defines a skew-symmetric algebroid structure on the bundle* $\tau : A(G) \to B$ *with the anchor*

$$a : A(G) \longmapsto \mathsf{T}B, \quad a = \mathsf{T}\beta|_{A(G)}.$$

**Proof.** The bracket $[\ ,\ ]_{\varepsilon}$ is obviously skew-symmetric and, for $f \in C^{\infty}(B)$, we have

$$\begin{aligned}[X, fY]_{\varepsilon}(p) &= [\overrightarrow{X}], \overrightarrow{fY}(p) = [\overrightarrow{X}, (f \circ \beta)\overrightarrow{Y}](p) \\ &= f(\beta(p))[\overrightarrow{X}], \overrightarrow{Y}](p) + \overrightarrow{X}(f \circ \beta)(p)\overrightarrow{Y}(p) \\ &= f[X, Y]_{\varepsilon}(p) + ((\mathsf{T}\beta(X))(f))(p)Y(p). \qquad \square\end{aligned}$$

**Theorem 19.** *The vector field* $\overleftrightarrow{X} = \overrightarrow{X} + \overleftarrow{X}$ *projects to* $a(X)$ *under both projections,* $\alpha$ *and* $\beta$, *and it is tangent to* $B$*-regarded as a submanifold of* $G$.

**Proof.** No difference with respect to the classical case. Let us mention only, that in the pre-algebroid case the vector field $\overleftrightarrow{X}$ is, in general, no longer multiplicative. This shows that the multiplicativity is not essential for the lifting procedures. □

**Corollary 3.** *The vector field* $\mathrm{d}_\mathsf{T}\overleftrightarrow{X}$ *is tangent to the submanifold* $A(G)$ *of* $\mathsf{T}G$.

Let us denote $\mathrm{d}_\mathsf{T}\overleftrightarrow{X}|_{A(G)}$ by $\mathrm{d}^G_\mathsf{T}X$ – the pre-Lie groupoid lift of a section $X$ of $A(G)$.

**Theorem 20.** *For each section* $X \in \otimes^1 A(G)$, *we have*

$$\mathrm{d}^G_\mathsf{T}X = \mathrm{d}^\varepsilon_\mathsf{T}X.$$

**Proof.** Since the set of functions $\{\iota(\mu) : \mu \in \otimes^1(\pi)\}$, where $\pi : A^*(G) \to B$ is the bundle dual to $\tau$, is a complete set of functions almost everywhere on $A(G)$, it is sufficient, in view of (76), to prove the equality

$$\mathrm{d}^G_\mathsf{T}X(\iota(\mu)) = \iota(\mathcal{L}^\varepsilon_X\mu) \quad \text{for all } \mu \in \otimes^1(\pi),$$

i.e.,

$$\mathrm{d}^G_\mathsf{T}X(\iota(\mu)) \circ Y = a(X)\langle\mu, Y\rangle - \langle\mu, [X, Y]_\varepsilon\rangle \tag{79}$$

for all $\mu \in \otimes^1(\pi)$, $X, Y \in \otimes^1(\tau)$.

We can find (at least locally) a function $f$ on $G$ vanishing on $B$ and such that $\mathrm{d}f$ represents $\mu$, i.e., $\langle\mu, Y\rangle = \langle \mathrm{d}f|_B, Y\rangle$. We have,

$$\begin{aligned}
\mathrm{d}^G_\mathsf{T}X(\iota(\mu)) \circ Y &= \langle \mathrm{d}_\mathsf{T}\overleftrightarrow{X}, \mathrm{d}\iota(\mu)\rangle \circ Y = \langle \mathrm{d}_\mathsf{T}\overleftrightarrow{X}, \mathrm{d}_\mathsf{T}\mathrm{d}f\rangle \circ Y = \mathrm{d}_\mathsf{T}\langle\overleftrightarrow{X}, \mathrm{d}f\rangle \circ Y \\
&= \iota(\mathrm{d}\,\langle\overleftrightarrow{X}, \mathrm{d}f\rangle) \circ Y = \langle \mathrm{d}\,\langle\overleftrightarrow{X}, \mathrm{d}f\rangle, Y\rangle = \langle \overrightarrow{Y}, \mathrm{d}\,\langle\overleftrightarrow{X}, \mathrm{d}f\rangle|_B \\
&= \overrightarrow{Y}(\langle\overleftrightarrow{X}, \mathrm{d}f\rangle)|_B = (\langle[\overrightarrow{Y}, \overleftrightarrow{X}], \mathrm{d}f\rangle + \overleftrightarrow{X}(\langle\overrightarrow{Y}, \mathrm{d}f\rangle))|_B \\
&= \langle[\overrightarrow{Y}, \overrightarrow{X}], \mathrm{d}f\rangle|_B + a(X)\langle Y, \mu\rangle \\
&= a(X)\langle Y, \mu\rangle - \langle\mu, [X, Y]_\varepsilon\rangle = \langle Y, \mathcal{L}^\varepsilon_X\mu\rangle.
\end{aligned}$$

Here we used the fact that $[\overrightarrow{Y}, \overleftarrow{X}]|_B = 0$. In general, the left and right prolongations do not commute as in the case of Lie groupoids, but they commute on $B$, and it is sufficient for our purposes. □

It is clear that relaxing the associativity assumption we get skew-algebroids as introduced in Section 1 which are no longer Lie algebroids.

**Example.** Let $V$ be a vector space and let $D : V \times V \to V$ be a skew-symmetric, bi-linear mapping. We define a pre-Lie group structure on $V$ (a pre-Lie grupoid over a single point) as follows. Let $X \star Y$ be the pre-group product given by

$$X \star Y = X + Y + D(X, Y).$$

It is clear that 0 is the unit element and $X \mapsto -X$ is the inverse mapping. $A(V)$ is canonically identified with $V$. We have the following formulae for prolongations:

$$\overrightarrow{X}(Z) = X + D(X, Z), \qquad \overleftarrow{X}(Z) = -X + D(X, Z),$$

and we obtain the bracket

$$[X, Y]_\varepsilon = [\overrightarrow{X}, \overrightarrow{Y}](0) = 2D(X, Y).$$

It is easy to see that this bracket satisfies the Jacobi identity if and only if '$\star$' is associative. Also the vector field $\overleftrightarrow{X}(Z) = 2D(X, Z)$ is multiplicative if and only if '$\star$' is associative.

In the above considerations concerning pre-Lie groupoids we implicitly assumed that the product $q_1q_2$ exists if and only if $\beta(q_1) = \alpha(q_2)$. However, relaxing the associativity assumption, we should probably change also this axiom as shown in the following example.

**Example.** Suppose that in a Lie group $D$ we have chosen a Lie subgroup $G$ and a 'complementary' submanifold such that
(1) $e \in M$ and $M^{-1} = M$,
(2) the composition $G \times M \ni (g, u) \mapsto gu \in D$ is a diffeomorphism,
(3) for each pair $u, v \in M$ there is a unique $g \in G$, denoted by $\varphi(u, v)$, such that $ugv \in M$.
We denote the element $u\varphi(u, v)v$ by $u \star v$.

For a given subgroup $G$ one can find such a triple $(D, M, G)$, at least locally, putting $M = \exp \mathfrak{m}$, where $\mathfrak{m}$ is a complementary subspace to the Lie algebra $\mathfrak{g}$ of $G$ in the Lie algebra $\mathfrak{d}$ of $D$. The reader easily recognizes some similarities with double Lie groups (e.g. if $M$ is a subgroup) and quasi-Poisson Lie groups (cf. [9] and the concept of quasi-triple in [1]).

Since every element $d \in D$ has unique decompositions $d = gu = vh$, where $g, h \in G$ and $u, v \in M$, we shall write $d = (g, u, v, h)$. We have obvious projections $\alpha, \beta : D \to G$,

$$\alpha(g, u, v, h) = g, \quad \beta(g, u, v, h) = h.$$

Now, we define a partial product in $D$ defined for pairs

$$d_1 = (g_1, u_1, v_1, h_1), \quad d_2 = (g_2, u_2, v_2, h_2)$$

such that

$$\beta(d_1)\varphi(u_1, u_2) = \varphi(v_1, v_2)\alpha(d_2)$$

by

$$d_1 \circ d_2 = (g_1, u_1 \star u_2, v_1 \star v_2, h_2).$$

This definition is correct, because

$$\begin{aligned} g_1(u_1 \star u_2) &= g_1u_1\varphi(u_1, u_2)u_2 = v_1h_1\varphi(u_1, u_2)u_2 \\ &= v_1\varphi(v_1, v_2)q_2u_2 = v_1\varphi(v_1, v_2)v_2h_2 = (v_1 \star v_2)h_2. \end{aligned}$$

Moreover, $\alpha(d_1 \circ d_2) = \alpha(d_1)$ and $\beta(d_1 \circ d_2) = \beta(d_2)$. Of course, this partial operation is not associative unless the $\star$-product is not associative. It is easy to see that elements $d = (g, e, e, g)$, i.e., $d = g \in G$, form the set of units and that every element $d = (g, u, v, h)$ has the inverse $i(d) = (h, u^{-1}, v^{-1}, g)$. If $M$ is also a subgroup, then what we get is exactly the Lie groupoid of the double group.

The question what are the algebroids being the infinitesimal versions of such structures we postpone to a separate paper.

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